Bernoulli's Equation

Reference: Landau & Lifshitz; Fluid Mechanics

1 Bernoulli's equation

1.1 General equations

Begin with the momentum equations

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0$$

Take scalar product with velocity

$$V_i \frac{\partial V_i}{\partial t} + V_i V_j \frac{\partial}{\partial x_j} V_i + \frac{1}{\rho} V_i \frac{\partial P}{\partial x_i} + V_i \frac{\partial \phi}{\partial x_i} = 0$$

Since $V^2 = V_i V_i$ then

$$\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + V_j \frac{\partial}{\partial x_j}\left(\frac{V^2}{2}\right) + \frac{1}{\rho} V_i \frac{\partial P}{\partial x_i} + V_i \frac{\partial \phi}{\partial x_i} = 0$$

First 2 terms are the derivative following the motion of $\frac{1}{2}V^2$ so that

$$\frac{d}{dt}\frac{V^2}{2} + \frac{1}{\rho} V_i \frac{\partial P}{\partial x_i} + V_i \frac{\partial \phi}{\partial x_i} = 0$$

Time independent flow:

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + V_i \frac{\partial P}{\partial x_i} = V_i \frac{\partial P}{\partial x_i}$$

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + V_i \frac{\partial \phi}{\partial x_i} = V_i \frac{\partial \phi}{\partial x_i}$$

Relationship between entropy, enthalpy and pressure

$$Tds = dh - \frac{1}{\rho} dP$$

$$T\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dP}{dt}$$

Hence for adiabatic flow:

$$\frac{1}{\rho}\frac{dP}{dt} = \frac{dh}{dt}$$

and this gives:

$$\frac{d}{dt}\left(\frac{1}{2}V^2\right) + \frac{dh}{dt} + \frac{d\phi}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}V^2 + h + \phi\right) = 0$$

so that, along a streamline:

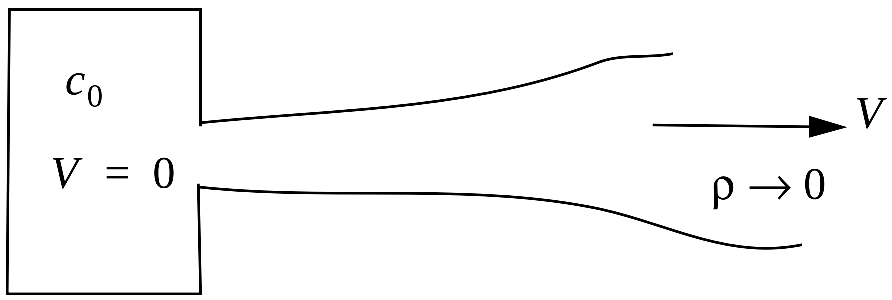
$$\frac{1}{2}V^2 + h + \phi = \text{constant} = \frac{1}{2}V_0^2 + h_0 + \phi_0$$

where the zero-subscripted quantities refer to value at some point alomg the streamline.

Expression for specific enthalpy:

$$h = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} = \frac{c_s^2}{\gamma - 1}$$

1.2 Application – expansion into a vacuum



Consider gas released from a box into a vacuum. As the gas expands, the density tends to zero at a long distance from the opening.

$$\rho \to 0 \Rightarrow c_s^2 = K\rho^{\gamma - 1} \to 0$$

Hence,

$$\frac{1}{2}V^2 = \frac{c_0^2}{\gamma - 1} \Rightarrow V = \left(\frac{2}{\gamma - 1}\right)^{1/2} c_0$$

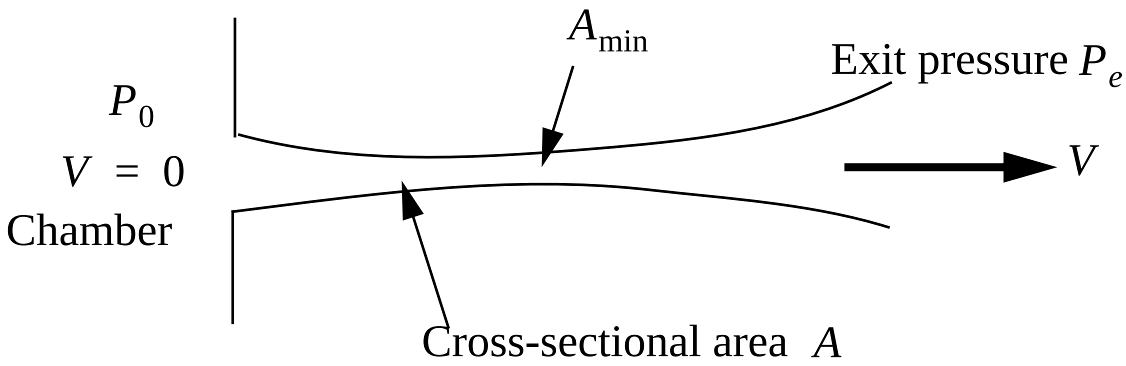
and the velocity is supersonic with respect to the original sound speed.

2 One dimensional steady flow

2.1 De Laval nozzle

Main object:

To study flows in nozzles of varying cross-section – in particular the De Laval nozzle.



The aim of the De Laval nozzle is to produce supersonic flow by reducing the exit pressure well below P_0. We consider a flow where the cross-sectional area changes slowly so that the flow can be considered one-dimensional.

Bernoulli's equation:

$$\frac{1}{2}V^2 + h = h_0$$

This implies that the maximum exit velocity is $\sqrt{2h_0}$.

Mass flux density:

$$j = \rho V$$

For steady flow

$$jA = \rho VA = \text{constant}$$

Variation of j along a streamline:

$$V^\beta \frac{\partial}{\partial x^\beta} V^\alpha = -\frac{1}{\rho}\frac{\partial P}{\partial x^\alpha}$$

$$V\frac{\partial V}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

Hence,

$$VdV = -\frac{1}{\rho}dP = -\frac{c_s^2}{\rho}d\rho$$

$$\Rightarrow \frac{d\rho}{dv} = -\frac{\rho V}{c_s^2}$$

The rate of change of the mass flux density wrt velocity is given by:

$$\frac{dj}{V} = \frac{d}{d\rho}(\rho V) = \rho + V\frac{d\rho}{dV} = \rho - \frac{\rho V^2}{c_s^2} = \rho\left(1 - \frac{V^2}{c_s^2}\right)$$

Keeping in mind that the object of the De Laval nozzle is to produce supersonic flow, we can see from this equation that the mass flux density increases until $V = c_s$ and then decreases.

Maximum value of mass flux density:

$$j_1 = \rho_1 c_1$$

where c_1 is the speed of sound where $V = c_1$.

Value of c_1

Use Bernouli's equation:

$$\frac{1}{2}V^2 + \frac{c_s^2}{\gamma - 1} = \frac{c_0^2}{\gamma - 1}$$

$$\frac{1}{2}c_1^2 + \frac{c_1^2}{\gamma - 1} = \frac{(\gamma + 1)}{2(\gamma - 1)}c_1^2 = \frac{c_0^2}{\gamma - 1}$$

$$\Rightarrow c_1 = \left(\frac{2}{\gamma + 1}\right)^{1/2} c_0$$

The following bit os only incidental to the properties of the De Laval nozzle. However, we can now use the above equation to obtain an expression for the maximum mass flux density $j_1 = \rho_1 c_1$. Since

$$c_s^2 = \gamma K \rho^{\gamma - 1} \quad \text{and} \quad c_0^2 = \gamma K \rho_o^{\gamma - 1}$$

then

$$\left(\frac{c_s}{c_0}\right)^2 = \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} \Rightarrow \frac{\rho}{\rho_0} = \left(\frac{c_s}{c_0}\right)^{\frac{2}{\gamma - 1}}$$

and, therefore

$$j_{\max} = \rho_1 c_1 = \rho_0 c_0 \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

2.2 Relationship between pressure & velocity

Use Bernoulli's equation again:

$$\frac{1}{2} V^2 + \frac{c_s^2}{\gamma - 1} = \frac{c_0^2}{\gamma - 1}$$

using the following expression for the speed of sound:

$$c_s^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} = c_0^2 \left(\frac{P}{P_0}\right)^{\frac{\gamma - 1}{\gamma}}$$

Hence,

$$\frac{1}{2} V^2 + \frac{c_0^2}{\gamma - 1} \left(\frac{P}{P_0}\right)^{\frac{\gamma - 1}{\gamma}} = \frac{c_0^2}{\gamma - 1}$$

$$\frac{P}{P_0} = \left[1 - \frac{(\gamma - 1)}{2} \frac{V^2}{c_0^2}\right]^{\frac{\gamma}{\gamma - 1}}$$

and this equation implies that the pressure decreases along the nozzle as V increases.

2.3 Relationship between mass flux density and pressure

Solve $P(V)$ equation for V:

$$\frac{V^2}{c_0^2} = \frac{2}{\gamma - 1}\left[1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma - 1}{\gamma}}\right]$$

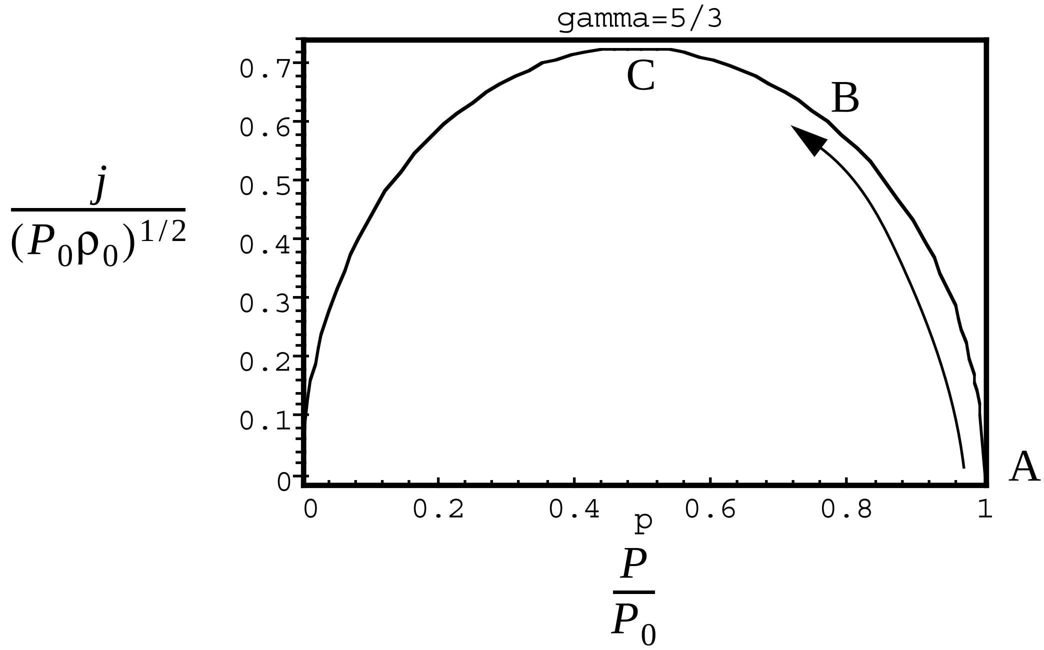
then

$$j^2 = \rho^2 V^2 = \frac{2}{\gamma - 1}\rho_0^2 c_0^2 \left(\frac{P}{P_0}\right)^{2/\gamma}\left[1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma - 1}{\gamma}}\right]$$

and since $c_0^2 = \dfrac{\gamma P_0}{\rho_0}$

$$j(P) = \left(\frac{2\gamma}{\gamma - 1}P_0\rho_0\right)^{1/2}\left(\frac{P}{P_0}\right)^{1/\gamma}\left[1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma - 1}{\gamma}}\right]^{1/2}$$

This relationship is of fundamental importance since it helps to define the minimum external pressure that is required to produce supersonic flow. The following plot helps to understand this.



Maple plot of mass flux density as a function of pressure for $\gamma = 5/3$

We have earlier seen that the mass flux density is a maximum at the sonic point. From the above plot, this occurs when $\frac{P}{P_0} \approx 0.5$. The initial chamber corresponds to the point A, thus the traversal of the nozzle correpsonds to traversing the above curve in the direction ABC, as shown. If the pressure only decreases to the value at point B, say, then j does not reach its maximum and the flow does not become supersonic. If, on the other hand, the pressure decreases to a value beyond point C, then j reaches a maximum, and the flow becomes supersonic. Any pressure beyond C correpsonds to an exit pressure (relative to P_0) which wil make the flow supersonic so that the pressure at C is the maximum exit pressure consistent with supersonic flow.