# Key Results from last lecture

$$\dot a^2 + kc^2 = \frac{8\pi G}{3}\rho a^2 \Rightarrow \frac{1}{c^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\rho a^2 - k$$

**Friedman Eq.**

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3c^2}\rho a^4 - ka^2, \quad \frac{dt}{d\eta} = \frac{a_0}{c}\frac{a}{a_0}$$

$$\left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 = \frac{k\Omega_0}{\Omega_0 - 1}\frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^4 - k\left(\frac{a}{a_0}\right)^2 \qquad \frac{a_0 H_0}{c}|\Omega_0 - 1|^{1/2} = |k|$$

**Boundary Condition**

$$\ddot a = -\frac{4\pi G}{3}(\rho + 3p)a$$

**G.R. Eq. II**

$$ds^2 = (cdt)^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

redefine as conformal time

$$\eta = c\int\frac{dt}{a(t)} \qquad d\Omega = d\theta^2 + \sin^2\theta d\phi^2$$

$$ds^2 = (a(\eta))^2\left[(d\eta)^2 - \left(\frac{dr^2}{1-kr^2} + r^2(d\Omega)\right)\right]$$

**Robertson-Walker Metric**

## Solutions for matter-dominated era and no cosmological constant

For convenience, use scaled variable

$$y = \frac{a}{a_0}$$

Relation between density and scale factor

$$\rho a^3 = \text{constant}$$
$$\Rightarrow \frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^3 = 1$$
$$\Rightarrow \frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^4 = \frac{a}{a_0} = y$$

Equation of State of Normal Matter

## Model Content of Universe by the Equation of State of the different forms of Matter/Energy

$$w_i \equiv \frac{P_i}{\rho_i} \qquad \rho_i \propto (\text{Volume})^{-(1+w_i)}$$

*e.g.,*

w=0 **for normal matter**

w=1/3 **for photons**

w=−1 **for Cosmological Constant**

$$\rho \propto V^{-1}$$

demo

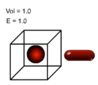
# Figuring Out the Equation of State

$$w \equiv \frac{p}{\rho c^2}$$

$$\rho_i a^{3w_i+3} = \text{constant}$$

$$\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^{3w+3} = 1$$

$$\left(\frac{\rho}{\rho_0}\right) = \frac{\left(\frac{M}{M_0}\right)}{\left(\frac{V}{V_0}\right)}$$




$$\frac{V}{V_0} = \left(\frac{a}{a_0}\right)^3$$

$$\frac{M}{M_0} = \frac{E}{E_0} = \left(\frac{a}{a_0}\right)^?$$

How does M/E goes as the scale factor?

$$3w + 3 = 3 - ?$$
$$w = -?/3$$

$$\left(\frac{\rho}{\rho_0}\right) = \frac{\left(\frac{a}{a_0}\right)^?}{\left(\frac{a}{a_0}\right)^3} \rightarrow \left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^{3-?} = 1$$

? = 0 for normal matter

? = -1 for photons

? = 3 for Cosmological Constant

# Flat Universe –Matter Dominated

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\rho a^2 - k$$

$$y \equiv \frac{a}{a_0}, \quad \frac{dy}{dt} = \frac{dy}{da}\frac{da}{dt} = \frac{1}{a_0}\frac{da}{dt}, \quad k = 0$$

1

$$\left(\frac{dy}{dt}\right)^2 = \frac{8\pi G\rho_0}{3H_0^2}H_0^2\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^2 = \frac{1}{\Omega_0}H_0^2\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^2$$ Friedman Equation for a flat Universe

$$\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^3 = 1$$ for matter dominated universe

$$\left(\frac{dy}{dt}\right)^2 = H_0^2\left(\frac{a}{a_0}\right)^{-1} = H_0^2 y^{-1}$$

$$\sqrt{y}dy = H_0 dt$$

$$\frac{2}{3}y^{3/2}dy = H_0 t$$

$$y = \left(\frac{3H_0 t}{2}\right)^{2/3}$$

This solution is the same as for the Newtonian case and implies the same age of the Universe, viz.

$$t_0 = \frac{2}{3}H_0^{-1} = 9.3\left(\frac{H_0}{70\,\mathrm{km\,s^{-1}\,Mpc}}\right)^{-1}\,\mathrm{Gyr}$$

This is less than the ages of the oldest stars in the Universe that we see in globular clusters

$$\frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^3 = 1$$

k=+1

In this case we use the conformal time substitution to solve the dual equations:

$$\left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 = \frac{k\Omega_0}{\Omega_0-1}\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^4 - k\left(\frac{a}{a_0}\right)^2$$

$$\left(\frac{dy}{d\eta}\right)^2 = \frac{\Omega_0}{\Omega_0 - 1}y - y^2$$
$$= \frac{\Omega_0^2}{4(\Omega_0-1)^2} - \left(y - \frac{\Omega_0}{2(\Omega_0-1)}\right)^2$$

We solve this equation in the following way. Make the substitution:

$$\begin{aligned} y &= \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta) \\ \Rightarrow \frac{dy}{d\eta} &= \frac{\Omega_0}{2(\Omega_0 - 1)}\sin\theta\frac{d\theta}{d\eta} \\ \left(\frac{dy}{d\eta}\right)^2 &= \frac{\Omega_0^2}{4(\Omega_0 - 1)^2}\sin^2\theta \\ \frac{d\theta}{d\eta} &= 1 \end{aligned}$$

Fr. Eq from last page

$$\left(\frac{dy}{d\eta}\right)^2 = \frac{\Omega_0^2}{4(\Omega_0-1)^2} - \left(y - \frac{\Omega_0}{2(\Omega_0-1)}\right)^2$$

$$\left(\frac{dy}{d\eta}\right)^2 = \frac{\Omega_0^2}{4(\Omega_0-1)^2} - \left(\frac{\Omega_0}{2(\Omega_0-1)}(1-\cos\theta) - \frac{\Omega_0}{2(\Omega_0-1)}\right)^2$$

Only way to reconcile these our substitution with our equation is for

The solution for y therefore is

$$y = \frac{a}{a_0} = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\eta)$$

$$\sin^2\theta = \frac{1}{2}\left(1 + \cos\frac{\theta}{2}\right)$$

Solution for t

$$\begin{aligned} \frac{dt}{d\eta} &= \frac{a_0}{c}y \\ &= \frac{a_0}{c}\frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\eta) \\ \Rightarrow t &= \frac{a_0}{c}\frac{\Omega_0}{2(\Omega_0 - 1)}(\eta - \sin\eta) \end{aligned}$$

We also know that

$$\frac{a_0 H_0}{c}(\Omega_0 - 1)^{1/2} = 1$$

Hence the solution for k=1, can also be expressed in the form:

$$\begin{aligned} a &= \frac{c}{H_0}\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}(1 - \cos\eta) \\ t &= \frac{1}{H_0}\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}(\eta - \sin\eta) \end{aligned}$$

Note that the timescale of this model Universe is still set by the *Hubble time* $H_0^{-1}$, but the numerical factor is different. The unit of length is set by the *Hubble length* $c/H_0$

# Flat Universe – Radiation Dominated

$$\left(\frac{dy}{dt}\right)^2 = H_0^2\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^2$$

$$\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^4 = 1 \text{ for radiation dominated universe}$$

$$\left(\frac{dy}{dt}\right)^2 = H_0^2\left(\frac{a}{a_0}\right)^{-2} = \frac{H_0^2}{y^2}$$

$$y\,dy = H_0\,dt$$

$$\frac{y^2}{2} = H_0 t$$

$$y = (2H_0 t)^{1/2}$$

# Flat Universe –Cosmological Constant Dominated

$$\left(\frac{dy}{dt}\right)^2 = H_0^2\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^2$$

$$\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^0 = 1 \text{ for cosmological constant dominated universe}$$

$$\left(\frac{dy}{dt}\right)^2 = H_0^2\left(\frac{a}{a_0}\right)^2 = H_0^2 y^2$$

$$\frac{1}{y}dy = H_0\,dt$$

$$\ln(y) = H_0 t$$

$$y = e^{H_0 t}$$

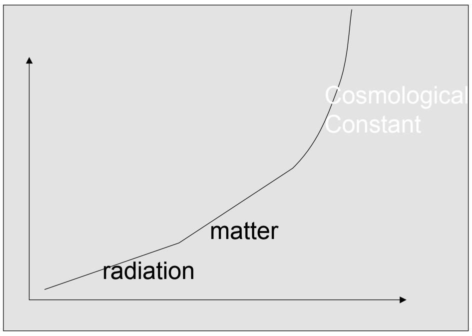
# Domination of the Universe

- As Universe Expands
  - Photon density decays as $a^4$
  - Matter density decays as $a^3$
  - Cosmological Constant density decays as $a^0$

$$\frac{\Omega_{rad}}{\Omega_M} = \frac{a}{a_0} = (1+z)$$

- Note that exactly flat Universe remains flat – i.e. $\Sigma\Omega_i=1$
- Cosmological Constant Models tend towards flatness overtime
- Other models tend away from flatness over time.

$$\frac{\Omega_\Lambda}{\Omega_M} = \left(\frac{a}{a_0}\right)^{-3} = (1+z)^{-3}$$


Cosmological
Constant
matter
radiation