Final Parametric Solution

$$\xi = \frac{1}{2}(1-\cos\eta) = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0}$$

$$\tau = \left(\frac{1}{2}\eta - \frac{1}{2}\sin\eta\right) = \frac{|1-\Omega_0|^{3/2}}{\Omega_0} H_0 t$$

A Quick Overview of Relativity

- Special Relativity:
  - The manifestation of requiring the speed of light to be invariant in all inertial (non-accelerating) reference frames
  - Leads to a more complicated view of the world where space and time have to be considered together – space-time

Minkowski Space

- Special relativity defined by cartesian coordinates, x^μ on a 4-dimensional manifold.
- Events in special relativity are specified by its location in time and space – a four-vector – e.g. V^μ

$$\left.\begin{aligned} x^0 &\equiv ct = t \\ x^1 &\equiv x \\ x^2 &\equiv y \\ x^3 &\equiv z \end{aligned}\right\} x^\mu$$

Minkowski Metric

- Metric tells you how to take the norm of a vector – e.g. the dot product.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In Minkowski space, the dot product of two vectors is, where we use the summation convention (lower and upper indices are summed over all possible values)

$$A \cdot B \equiv \eta_{\mu\nu} A^\mu B^\nu = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$$

Minkowski Metric

$$ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

Time

- Note that for a particle with fixed coordinates, $ds^2 = -dt^2 < 0$

Define Proper time as $d\tau^2 \equiv -ds^2$

The proper time elapsed along a trajectory through spacetime represents the actual time measured by the observer.

Tensors

- General Relativity requires curved space
- Use Tensors – which are a way of expressing information in a coordinate invariant way. If an equation is expressed as a tensor in one coordinate system, it will be valid in all systems.
- Tensors are objects like vectors, or matrices, except they may have any range of indices and must transform in a coordinate invariant way.

The Metric Tensor

The metric tensor in GR is the foundation of the subject. It is the generalisation of the Minkowski metric.

It describes spacetime in a possibly a non-flat, non-cartesian case.

e.g. Spherical coordinate flat case

$$\left.\begin{aligned} x^0 &= t \\ x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\sin\theta \end{aligned}\right\} x^\mu$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

An Example –Maxwell's equations

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

and the electric charge density ρ and current $\vec{J}$ into a four-vector J^μ:

$$J^\mu = (\rho,\ \vec{J}).$$

In this notation, Maxwell's equations

$$\begin{aligned} \nabla\times\mathbf{B} - \partial_t\mathbf{E} &= 4\pi\mathbf{J} \\ \nabla\cdot\mathbf{E} &= 4\pi\rho \\ \nabla\times\mathbf{E} + \partial_t\mathbf{B} &= 0 \\ \nabla\cdot\mathbf{B} &= 0 \end{aligned}$$

shrink into two relations,

$$\begin{aligned} \partial_\mu F^{\nu\mu} &= 4\pi J^\nu \\ \partial_{[\mu}F_{\nu\lambda]} &= 0. \end{aligned}$$

These are true in Minkowski space, but the generalization to a curved spacetime is immediate; just replace $\partial_\mu \to \nabla_\mu$:

$$\begin{aligned} \nabla_\mu F^{\nu\mu} &= 4\pi J^\nu \\ \nabla_{[\mu}F_{\nu\lambda]} &= 0. \end{aligned}$$

These equations govern the behavior of electromagentic fields in general relativity.

The basic components of GR equations

- Riemann curvature tensor – which is a complicated expression derived from the metric tensor –gives curvature
- Geodesics –the shortest space-time distance between two points. Imagine a set of paths is parameterised by a single parameter λ

$$ds = \sqrt{\left|g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right|}\,d\lambda\,.$$

- Test particles move along geodesics

Einstein's Equation of motion

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Lefthand side describes curvature of space time

$T_{\mu\nu}$ is called the stress-energy Tensor and contains a complete description of energy and momentum of all matter fields.

Lots of information in the $T_{\mu\nu}$ and $G_{\mu\nu}$ - end up with very complicated linked highly non-linear equations.

GR and Cosmology

- Robertson Walker Metric (independent of Einstein's Equations) Provides a completely descriptive metric for a homogenous and isotropic universe

$$ds^2 = (cdt)^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

- r,θ,φ are spherical coordinates
- a(t) describes the size of a piece of space over time
- k tells you the curvature (-1,0,1) -> (open, flat, closed)

Einstein's equation for a perfect fluid

- a is the scale factor – tracks a piece of the Universe.
- p and ρ are pressure and density of matter (perfect fluid)

$$\boxed{\begin{aligned} \dot{a}^2 + kc^2 &= \frac{8\pi G}{3}\rho a^2 \\ \ddot{a} &= -\frac{4\pi G}{3}(\rho + 3p)a \end{aligned}}$$

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{c^2}\frac{8\pi G}{3}\rho$$

divide 1st eq. by (ac)^2

$$\frac{k}{a^2} = \frac{1}{c^2}\left(\frac{8\pi G}{3}\rho - \left(\frac{\dot{a}}{a}\right)^2\right)$$

move k/a^2 to 1 side, factor out c^2

$$\frac{k}{a^2} = \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{8\pi G}{3}\rho\left(\frac{\dot{a}}{a}\right)^{-2} - 1\right)$$

factor out (adot/a)^2

$\frac{\dot{a}}{a} = H_0$ and ρ_{crit} as before

$$\frac{k}{a^2} = \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\rho}{\rho_{crit}} - 1\right)$$

Summary of Behaviour of Density parameter and Geometry

$$\begin{array}{l} \text{FLAT} \\ \text{OPEN} \\ \text{CLOSED} \end{array} \left\{ \begin{array}{lll} \Omega_0 = 1 & k = 0 & \Omega(t) = 1 \\ \Omega_0 < 1 & k = -1 & \Omega(t) < 1 \\ \Omega_0 > 1 & k = +1 & \Omega(t) > 1 \end{array} \right\} \text{for all time}$$

Contributions to Density

$$\Omega_0 = \sum \frac{\rho_{i,0}}{\rho_{crit,0}} = \sum \Omega_{i,0}$$

$$\Omega_0 = \Omega_{M,0} + \Omega_{\gamma,0} + \Omega_{\upsilon,0} + \Omega_{\Lambda,0} + \Omega_{?,0}$$

Critical Density Value.

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G}$$

$$\rho_{c,0} = 9.2x10^{-27} kg/m^3 \left(\frac{H_0}{70km/s/Mpc} \right)^2$$

$$\rho_{c,0} = 1.4x10^{-7} M_{sun}/pc^3 \left(\frac{H_0}{70km/s/Mpc} \right)^2$$

Contributions to Ω

- Within a Megaparsec (Mpc) There are two large galaxies - Andromeda and Milky Way - 10^{12}solar masses of material - So density is roughly 10^{-6}solar Masses per cubic parsec.
- Radiation from Big Bang has approx Ω_{rad}=5x10^{-5}
- Current Concordance Model of Cosmology has
  - $\Sigma\Omega_i$=1.00
  - Ω_M=0.27
  - Ω_Λ=0.73
  - $\Omega_{\text{everything else}}$<0.01

Solving Einstein's Equations

Robertson-Walker Metric

$$ds^2 = (cdt)^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$$

redefine as conformal time

$$\eta = c\int \frac{dt}{a(t)} \qquad d\Omega = d\theta^2 + \sin^2\theta d\phi^2$$

$$ds^2 = (a(\eta))^2 \left[(d\eta)^2 - \left(\frac{dr^2}{1-kr^2} + r^2(d\Omega) \right) \right]$$

Friedman Equation

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{c^2}\frac{8\pi G}{3}\rho$$

re-writing 1st G.R. eq

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\rho a^2 - k$$

re-arranging above and moving out a^2

Relation between different time coordinates

$$\eta = c\int \frac{dt}{a(t)}$$

$$\begin{aligned} cdt &= a(t)d\eta \\ \Rightarrow \frac{dt}{d\eta} &= \frac{a(t)}{c} \\ \frac{1}{c}\frac{d\eta}{dt} &= \frac{1}{a} \end{aligned}$$

The Friedmann equation is transformed by substituting:

$$\frac{1}{c}\frac{da}{dt} = \frac{da}{d\eta} \times \frac{1}{c}\frac{d\eta}{dt}$$

chain rule for differentiation

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\rho a^2 - k \qquad = \frac{1}{a}\frac{da}{d\eta}$$

replacing dn/dt with relation above.

so that it can be written:

replace (1/c da/dt) with above expression. move a^2 over

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3c^2}\rho a^4 - ka^2$$

Normalise equation to current epoch

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3c^2}\rho a^4 - ka^2$$

$$y \equiv \frac{a}{a_0} \qquad \Omega_0 = \frac{8\pi G}{3H_0^2} \qquad \frac{d}{d\eta}\left(\frac{a}{a_0}\right) = \frac{1}{a_0}\frac{da}{d\eta}$$

$$\left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 a_0^{\ 2} = \frac{\Omega_0 H_0^2}{\rho_0 c^2}\rho\left(\frac{a}{a_0}\right)^4 a_0^4 - k\left(\frac{a}{a_0}\right)^2 a_0^2$$

$$\left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 = \frac{a_0^2\Omega_0 H_0^2}{c^2}\frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^4 - k\left(\frac{a}{a_0}\right)^2$$

Boundary condition

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{c^2}\frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad H_0 = \frac{\dot{a}}{a_0}(\text{now})$$

$$\frac{1}{c^2}(H_0)^2 = \frac{1}{c^2}\frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^{\ 2}} \quad \text{at current epoch}$$

Which at current epoch gives

$$a_0^2 \frac{H_0^2}{c^2} = a_0^2 \left(\frac{8\pi G \rho_0}{3c^2} - \frac{k}{a_0^2} \right)$$

previous equation with a0^2 mult through

$$\frac{a_0^2 H_0^2}{c^2} = \frac{a_0^2 H_0^2}{c^2}\Omega_0 - k$$

replace in Omega_0 get rid of a0^2 in denominator

$$\Rightarrow \frac{a_0^2 H_0^2}{c^2}(\Omega_0 - 1) = k$$

$$\frac{a_0 H_0}{c}|\Omega_0 - 1|^{1/2} = |k|$$

Hence a_0 is determined by the Hubble constant, the density parameter and the geometry of the Universe but only when the density parameter is not unity

Radius of Curvature of the Universe

Final form of Friedmann equations

$$k = \Omega_0 - 1 = 0 \quad \left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 = \frac{a_0^2 H_0^2}{c^2}\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^4$$

$$k = \pm 1 \quad \left[\frac{d}{d\eta}\left(\frac{a}{a_0}\right)\right]^2 = \frac{k\Omega_0}{\Omega_0 - 1}\left(\frac{\rho}{\rho_0}\right)\left(\frac{a}{a_0}\right)^4 - k\left(\frac{a}{a_0}\right)^2$$

In all cases: $\quad \frac{dt}{d\eta} = \frac{a(\eta)}{c} = \frac{a_0}{c}\left(\frac{a}{a_0}\right)$

Matter dominated era, with zero cosmological constant

In the present era of the Universe, matter dominates radiation. If we also assume that the cosmological constant is zero, then the above equations can be solved quite straightforwardly.