

Uncertainties

Why do they matter?

Two Quotes:

The uncertainty is almost more important than the number itself.

(ANU Physics Researcher, 2010)

Uncertainty is that stupid number you calculate to keep your demonstrator happy.

(ANU Physics Student, 2010)

Which is true?

Example 1

Imagine that you work for a car company. You are designing the doors of a new car model. You want the door to fit snugly into the door frame, so that it doesn't rattle, water can't get in, and to minimise wind drag.



The door is supposed to be 1200 mm high. But the manufacturers cannot guarantee that every door and every door frame they produce is exactly 1200 mm high. There will be an uncertainty in this height. It might depend on how worn down the stamping machines are, how pure the steel is, how hot the shop floor was on the day of manufacture, and many other things.

Let's say you get the following quotes from different manufacturers, to make a door frame and a door.

1. Frame size $1200 \pm 10\text{mm}$, door size $1199 \pm 10\text{mm}$, cheap
2. Frame size $1200 \pm 1\text{mm}$, door size $1199 \pm 1\text{mm}$, average price

3. Frame size $1200 \pm 0.5\text{mm}$, door size $1199 \pm 0.5\text{mm}$, expensive
4. Frame size $1200 \pm 0.1\text{mm}$, door size $1199 \pm 0.1\text{mm}$, very expensive

Which quote would you go for?

If you went for the first quote, you could have a frame size anywhere in the range $1190 - 1210\text{mm}$ and a door with a size in the range $1189 - 1209$. So you could easily have a door larger than the frame (which wouldn't fit) or at the opposite extreme, a door 21mm smaller than the frame - such a big gap that water would almost certainly get through. Not a good quote to accept.

If you went with the second quote, you could still have a frame that was smaller than the door (if a particular frame had a below average size and the particular door had an above average size). Most of the time it would work but you'd have to throw out a certain fraction of doors and frames.

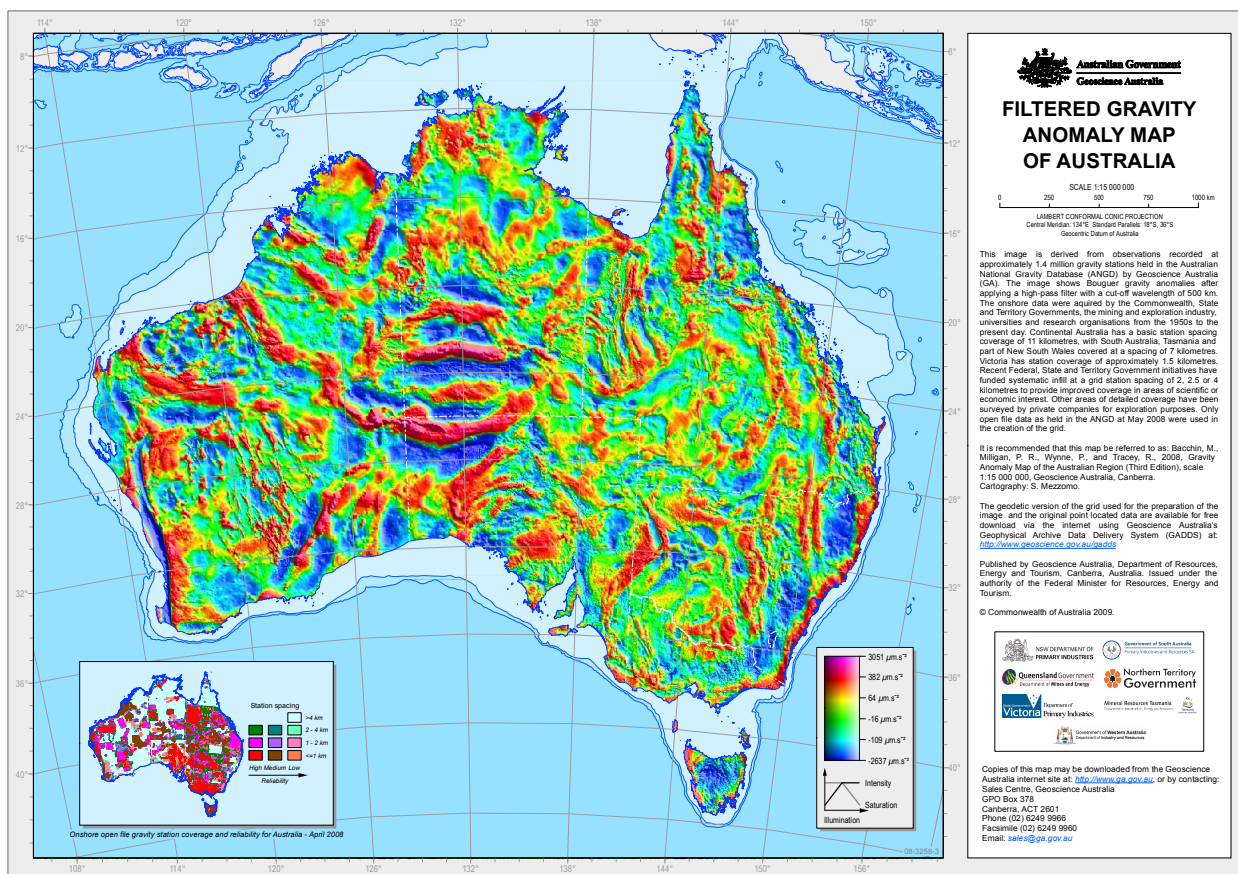
If you went with the third quote, the smallest frame you'd get would have a size of $1200 - 0.5 = 1199.5\text{mm}$ and the largest door you'd get would have the same size - so you're OK, they would always fit. So this would be acceptable. But you might end up with a 2mm gap, which might make the door rattle a bit.

With the last quote, the biggest gap would be 1.2mm. If the manufacturers uncertainties are believable, you could get away with a door size of 1199.9mm, in which case your gap would never be larger than 0.2mm, which would be very snug. Perhaps an option for luxury cars?

So - changing the uncertainties can make a given set of components usable or unusable,

Example 2

One way of finding buried metal deposits is to measure the gravity at the surface of the Earth. As metal deposits are denser than other rocks, gravity is a little greater when you are directly over them.



Let's say that you suspect that there might be iron ore at a given location. You make a gravity measurement to find out. If iron ore is present, you expect gravity to increase by $> 4\mu\text{m.s}^{-2}$.

Would you go to the great expense of doing some sample drilling if you measured an increase in gravity of:

1. $5 \pm 100\mu\text{m.s}^{-2}$?
2. $5 \pm 2\mu\text{m.s}^{-2}$?
3. $5 \pm 0.1\mu\text{m.s}^{-2}$?

If you got the first measurement, you basically haven't learned anything - your data are perfectly consistent with iron being present, but are quite consistent with nothing being present. The change in gravity could lie anywhere in the range $-95 - 105\mu\text{m.s}^{-2}$. If you got the last measurement you can be pretty sure iron is present, as the whole range in which the data can lie ($4.9 - 5.1\mu\text{m.s}^{-2}$) is greater than $4\mu\text{m.s}^{-2}$. The middle measurement tells you that the gravity change lies in the range $3 - 7\mu\text{m.s}^{-2}$, so there is a chance it lies below $4\mu\text{m.s}^{-2}$, but it's somewhat more likely it lies above this value.

Thus getting the same answer ($5\mu\text{m.s}^{-2}$) can lead to quite different conclusions depending on the uncertainties.

Conclusion

Even if you get the same answers, the conclusions you draw from them will depend on the uncertainties.

How To Determine Uncertainties

There is no magic formula for this. You have to think through how you are making your measurement or building your components, think of what could vary, and by how much.

Example: Bolts

Let's imagine you are making steel bolts on a computer controlled milling machine. Why might their radius vary?

- The steel will vary in its exact composition from batch to batch. Stronger batches will probably resist the milling process and end up slightly larger.
- The temperature of the bolts being milled will vary with time of day, season and how long the milling has been going on, which will cause the metal to expand. If cut to a given size when expanded, the size at room temperature will vary.
- The voltage of the power supply controlling the milling robot will change, causing small changes in where it positions the milling tool.
- The blade used in the milling will slowly wear down, and periodically be replaced. This will cause a periodic change in size.
- Air pressure, humidity, oil temperature and composition, cutting blade composition, hydraulic pressure, vibrations caused by other equipment in the factory, differences between different milling machines, the slow plastic deformation of the milling machine components all introduce more variation.

Example: Measuring height

Let's say you are trying to measure whether one person is short enough to be in a particular sporting category. Why might there be an uncertainty in their height?

- It's hard to read a height off a ruler to better than about half the smallest tick mark.
- Rulers are not all precisely the same length, due to variations in how they are measured.
- The length of a given ruler will change due to temperature (expanding when it is hot).
- The height of a person varies slightly with fluid consumption, posture and time of day.
- Different parts of the head will be highest depending on posture.
- If you are using a ruler, you will need to assume that the floor is horizontal, and have some horizontal object to take the height from the head to the ruler. Both will probably not be precisely horizontal.

In both cases, you could decrease your uncertainties by use of better (more expensive) equipment, controlling the temperature, changing parts more regularly, making all observations at the same time of day etc. Whether you need to will depend on what your purpose was in making the measurement, and how small an uncertainty is needed to let you achieve your purpose.

Scatter in Repeat Measurements

The best way to measure your uncertainty is to make repeat measurements, and see how much they vary.

Example - let's a factory sends you ten bolts, claiming that they should have a length of $450 \pm 20\text{mm}$. You measure the length of these bolts, using a very accurate method (uncertainty of less than 1mm). Here are the lengths you measure:

476.3 mm
493.9 mm
412.9 mm
423.7 mm
430.0 mm

439.1 mm
456.1 mm
499.2 mm
434.0 mm
435.7 mm

Do you believe the factory's uncertainty value?

No - clearly lots of them lie outside the range $430 - 470\text{mm}$. The average value you measure is indeed 450, but the range goes from 412 up to 499, so a better description of these bolts would be a length of $450 \pm 50\text{mm}$.

Warning

This "scatter in repeat measurements" technique is usually the best way to determine uncertainties. In the real world, people nearly always underestimate their uncertainties, and looking at the scatter is an excellent check.

It is, however, time-consuming. And for it to really work, the repeat measurements must be **independent**. What does this mean? It means that all the possible sources of variation are different between the different measurements. Say, for example you measure someone's height ten times. If the measurements were all made at the same time of day, using the same ruler, then any uncertainty due to different rulers or changes in height during the day will not be seen, and you will underestimate the true uncertainty.

How to Quote Uncertainties

Any of the methods below are acceptable.

Method 1:

State the range in which the number must lie.

e.g. "Our model predicts that global average temperatures will rise by between 2.3 and 5.4 degrees by 2100 AD".

Method 2:

Give a best estimate and a plus-or-minus range

e.g. "The prime-minister's approval rating was $23 \pm 3\%$ "

Method 3:

It can often be the case that the uncertainties in one direction are greater than another. In this case, quote your best estimate value and different positive and negative uncertainties.

e.g. "The expansion rate of the universe is $72^{+3.4}_{-2.3}\text{kms}^{-1}\text{Mpc}^{-1}$ "

Significant Figures

If the uncertainties really matter, you should always use one of the above methods to quote it. But sometimes an accurate knowledge of the uncertainty is not so important, and it's not worth the bother of calculating an accurate value. In this case, you should use the number of significant figures to tell readers what the uncertainties are.

Imply the uncertainty by the number of significant figures quoted. The rule is:

“The second last significant figure should be certain - the last one can be uncertain.”

So if you say that the mass of a ball is 34.52 kg, you know that the 34.5 is accurate, but that the final 2 is less certain. This could correspond to 34.52 ± 0.01 or 34.53 ± 0.04 , but if the true uncertainty was more like 34.52 ± 0.1 , it should only have been quoted as 34.5, as the 5 is now uncertain. If the uncertainty is something like 34.52 ± 0.07 , things are ambiguous - either 34.52 or 34.5 would be acceptable.

One common mistake is to quote FAR too many significant figures. If, for example, you are trying to calculate the average speed of a runner who ran the 100 metres sprint in 8.24 seconds, your calculator will tell you that $100/8.24=12.13592233009709$ m/s. But if you write this down, people might think that it really was measured this accurately, which it certainly wasn't. The last eight or so digits are fantasy, and it is wasteful and misleading to write them down.

Propagation of Uncertainty: Simple Method

A common situation - you know the uncertainty in the things you actually measure, but what you want to know is the uncertainty in some final result which is derived from these measured quantities, perhaps by a complicated equation.

To put it mathematically, you have some function $F(x,y,...)$ which you want to know, which depends in some way on the variables $x, y, ...$. If you know the uncertainties in $x, y, ...$, what is the uncertainty in $F(x,y,...)$?

In this section we present a simple way to deal with situations like this. Later in the course I will show a more sophisticated way (which only sometimes works).

All you do is calculate the answer repeatedly, varying the measured quantities to cover the range of their uncertainty. And see how much the final result changes. This gives you the uncertainty in the final result.

Example

Let us imagine that you are working at the Australian Institute of Sport, trying to use video footage to measure the speed at which tennis players serve. You want to try out different serve techniques and see which ones improve the speed of service.

From the video footage you measure four things:

P_1 - the position at which the ball leaves the racquet

T_1 - the time at which the ball leaves the racquet

P_2 - the position at which the ball touches the ground on the other side of the court

T_2 - the time at which the ball touches the ground on the other side of the court.

There is an uncertainty in each of these - you can't measure the positions to better than a pixel, and you can't measure the times to better than a frame in the video.

How accurately can you measure the **speed**?

The average speed is

$$s = \frac{P_2 - P_1}{T_2 - T_1}$$

(that's just the definition of average speed - distance covered divided by the time taken to do it). So our function $F(P_1, P_2, T_1, T_2)$ is the equation above, and the variables upon which it depends are P_1 , P_2 , T_1 and T_2 .

But what is the uncertainty in this average speed?

Let's use these numbers:

$$P_1 = 0.5 \pm 0.1m$$

$$P_2 = 22.1 \pm 0.1m$$

$$T_1 = 1.0 \pm 0.05s$$

$$T_2 = 2.3 \pm 0.05s$$

The average speed is thus

$$s = \frac{P_2 - P_1}{T_2 - T_1} = \frac{22.1 - 0.5}{2.3 - 1.0} = \frac{21.6}{1.3} = 16.62ms^{-1}$$

but what is the uncertainty in this?

Just try varying the numbers by the uncertainties and recalculating. In this case, the highest velocity will be if the distance is larger (P_2 larger and P_1 smaller) and time smaller (T_2 smaller and T_1 larger) - i.e. if

$$P_1 = 0.5 - 0.1 = 0.4m$$

$$P_2 = 22.1 + 0.1 = 22.2m$$

$$T_1 = 1.0 + 0.05s$$

$$T_2 = 2.3 - 0.05s$$

$$s = \frac{P_2 - P_1}{T_2 - T_1} = \frac{22.2 - 0.4}{2.25 - 1.05} = \frac{21.8}{1.2} = 18.17ms^{-1}$$

and the smallest velocity will be if the distance is smaller and the time larger - i.e.

$$s = \frac{P_2 - P_1}{T_2 - T_1} = \frac{22.0 - 0.6}{2.35 - 0.95} = \frac{21.4}{1.4} = 15.28ms^{-1}$$

So we know that the speed s will lie in the range $15.28 - 18.17ms^{-1}$. So it can be up to $16.62 - 15.28 = 1.34ms^{-1}$ below the best estimate, and up to $18.17 - 16.62 = 1.55ms^{-1}$ above the best estimate.

Notice that the upward uncertainty is greater than the downward one. A result like this is written as:

$$s = 16.62^{+1.55}_{-1.34} ms^{-1}$$

Outlying Points and the Bell Curve

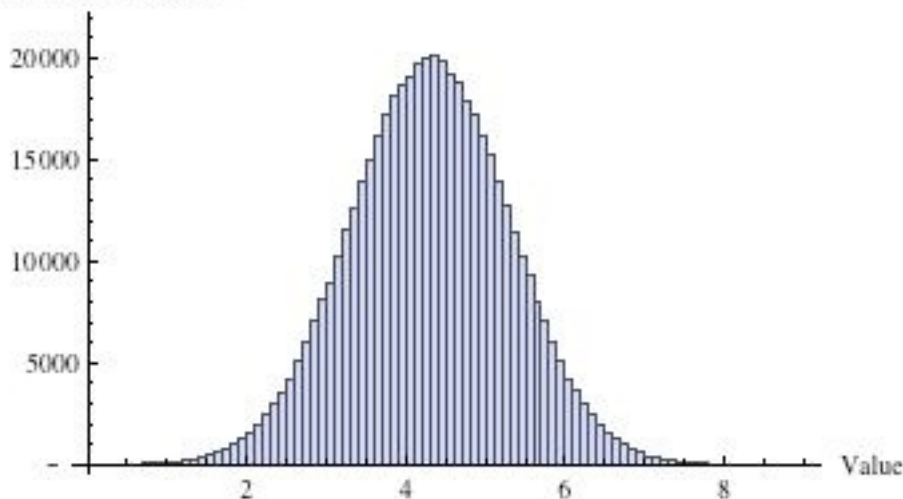
In all the above discussions, we've been assuming that an uncertain number has a well defined range. So if some number is 10 ± 2 , it will always lie in the range $8 - 12$.

Most real-world situations do not work like this. You typically find most observations close to the mean value, but a small number of observations can be very far away.

These way-out numbers can be genuine stuff-ups. If ten people in your lab measure a particular value and get answers in the range 4-7, but the last person gets 124, odds are that last person stuffed up and needs to check their working or re-do the experiment.

But even if you exclude stuff-ups, you will occasionally find values very far from the mean. Here is a typical histogram:

Number of Measurements



Notice that most measurements line fairly close to the mean (around 4.4 in this case). Around 90% of measurements lie in the range 3-6, but there are a small number further out.

This sort of curve is called a bell curve. The most famous example is the Gaussian Function:

$$p(x) \propto e^{-\frac{(x - \bar{x})^2}{2\sigma^2}}$$

where $p(x)$ is the probability of seeing a particular value of x , \bar{x} is the mean value of x , and we'll come back to σ .

Notice that while the probability gets pretty small of seeing a value very different from the mean, this probability is never zero.

How does this happen? You get a Gaussian curve if you assume that your result comes from adding together many random processes. For example, the size of a bolt might depend on temperature, pressure, wear on the cutter and many other things. And if you assume that no single process dominates the uncertainty, and that all the different factors are independent.

Each of these factors can either increase or decrease the length of an individual bolt. Usually some will increase it and some will decrease it, partially cancelling out. This is why most bolts will have a length close to the mean. But occasionally most of the factors will have the same affect and give you a very large or a very small bolt.

The Standard Uncertainty

This bell curve is a problem. How can you specify how uncertain something is? The range of a bell curve is infinite - if you keep making enough measurements, you will eventually get an incredibly large or small one. So the range of any sufficiently well studied process will be infinite! But these outlying measurements are very rare - most of the time you will get measurements close to the mean.

People have thought of various ways of measuring uncertainty which deal with these rare outlying points. You can, for example, use the interquartile range (the range of values within which 75% of the data points will lie).

But by far the most commonly used way to measure the range is the standard uncertainty (also known as the standard deviation), usually written σ (sigma).

How is this defined? Say you measure some parameter x a number n times. The first measurement is x_1 , the second is x_2 and so on. You work out the mean value \bar{x} by adding up all the measurements and then dividing by the number of measurements, i.e:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{1}{n} \sum_{i=1}^n x_i$$

You can then work out the standard deviation s by taking each measurement, working out how far it is from the mean, squaring all these values, dividing by n and taking the square root.

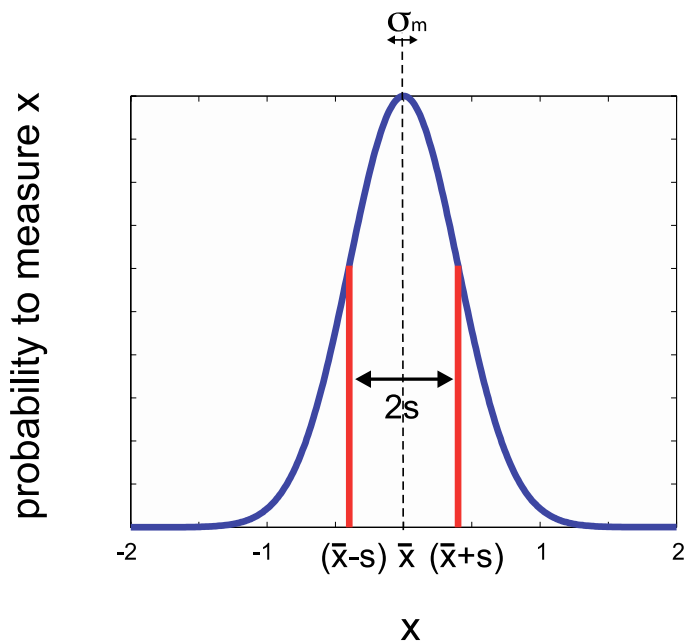
$$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Most calculators and maths/stats computer packages will help you calculate this.

What does a standard uncertainty mean?

If the uncertainties follow a Gaussian distribution, then you expect 68% of the data points to lie within one standard uncertainty of the true value, 95% to lie within two standard uncertainties, and 99.7% to lie within three standard uncertainties.

In practice, many sets of data do not follow a Gaussian distribution particularly well. Often there are more weirdo way-out points than such a distribution would predict. So use this with caution.



So this is rather different from the range of the data. If someone quotes a standard uncertainty, you expect to find quite a few points outside this range.

Quadrature

What happens if your result depends on lots of uncertain things? For example, let's say you have a pile of 10 bricks, each brick of size $10 \pm 0.1 \text{ cm}$. What is the size of the whole pile, and the uncertainty in it?

The most likely size of the whole pile is just 100 cm, but what is the uncertainty? It could be that all ten bricks are smaller than average (so pile height is $10 \times (10 - 0.1) = 99 \text{ cm}$) or are all larger than average (101 cm). But if the size of the different bricks are independent of each other, odds are some will be larger and some smaller than the average.

This means that if you add up a lot of independent random things, odds are some of the random effects will cancel out, and the final uncertainty will not just be the sum of the effects of the individual random things.

If you assume that the uncertainties follow a Gaussian distribution, are independent of each other, then you can show that the correct way to add up uncertainties is to square them, add the squared values, and then take the square root. This is known as “adding in quadrature”.

To put it mathematically, if $X = A + B + C + \dots$, and the uncertainty in A is σ_A , etc, then:

$$\sigma_X = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \dots}$$

This means that the uncertainty in X is more than the uncertainty in each of the individual components, but less than the sum of their uncertainties. This kind of makes sense - adding a new uncertainty should always make things less certain, but because different

uncertain factors may go in opposite directions, you would expect them to cancel out some fraction of the time.

Averaging Repeat Measurements

A special case of quadrature is when you have a whole string of measurements, each with the same uncertainty, and you average them. Let's say you have n individual measurements each with an uncertainty σ . What is the uncertainty in the average a ?

The average is just the sum of the n measurements, divided by n , so using the quadrature equation above, you find that:

$$a = \frac{\sigma}{\sqrt{n}}$$

Actually, this is almost right, but for complicated reasons the correct equation is

$$a = \frac{\sigma}{\sqrt{n-1}}$$

though this gives the same answer (close enough) for large values of n .

So what this means is that if you average a whole series of independent measurements of some quantity, the *mean* of these measurements has a standard uncertainty that is less than that of the individual measurements. The uncertainty in the mean goes down roughly as the square root of the number of independent measurements you make.

Worry about the biggest uncertainty

One consequence of adding uncertainties in quadrature is that if you add two uncertainties and one is a little larger than the other, the larger one will completely dominate the total final uncertainty.

Thus you should always try to find the biggest source of uncertainty and reduce it. Working hard to reduce more minor sources of uncertainty will have little effect on the final result.

More Sophisticated Way of doing Uncertainty Propagation

Remember - uncertainty propagation is the art of finding the uncertainty in some functions $F(x,y,...)$, from the uncertainties in the numbers that make it up, $x,y,....$

The simple method works well - just plug in numbers spanning the range of input uncertainties and see how much the output varies.

But there is a more sophisticated way. It only works if the following conditions are met:

- The different input variables are independent of each other
- The uncertainties follow a bell-curve distribution

- The uncertainties are small compared to the mean values.

But if you make these assumptions, and use the concept of adding in quadrature as discussed above, you can work out the uncertainty using the following equations.

In all these equations, σ_X is the standard uncertainty in X , σ_A is the standard uncertainty in A , and so on.

sum or difference - use the absolute uncertainties:

If $X = A + B$ or $X = A - B$ then

$$(\sigma_X)^2 = (\sigma_A)^2 + (\sigma_B)^2$$

product or fraction - use the relative uncertainties:

If $X = A \times B$ or $X = A/B$ then

$$\left(\frac{\sigma_X}{X}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2$$

Adding a constant

If $X = A + C$, where C is a constant with negligible uncertainty, then

$$\sigma_X = \sigma_A$$

Multiplying by a Constant

If $X = C \times A$, where C is a constant with negligible uncertainty,

$$\sigma_X = C \sigma_A$$

Raising to a Constant power

If $X = A^n$, and the uncertainty in n is small enough to ignore, then

$$\frac{\sigma_X}{X} = n \frac{\sigma_A}{A}$$

Logarithms

If $X = \ln(A)$ (log to the base e), then:

$$\sigma_X = \frac{\sigma_A}{A}$$

Exponents

If $X = e^A$, then:

$$\frac{\sigma_X}{X} = \sigma_A$$

General Rule

All the above equations, and many more, can be deduced from the following general rule. The general rule for calculating the uncertainty of any function of individually measured values $X = f(A, B, C, \dots)$ is

$$(\sigma_X)^2 = (\sigma_{X,A})^2 + (\sigma_{X,B})^2 + (\sigma_{X,C})^2 + \dots$$

where

$$\sigma_{X,A} = \frac{\partial X}{\partial A} \sigma_A$$

and so on. This uses partial differentiation, which you may not yet be familiar with. Don't worry if it makes no sense – you can just use above simpler equations.

Worked Example

Let's go back to the tennis player example we discussed above.

“Let us imagine that you are working at the Australian Institute of Sport, trying to use video footage to measure the speed at which tennis players serve. You want to try out different serve techniques and see which ones improve the speed of service.

From the video footage you measure four things:

P_1 - the position at which the ball leaves the racquet

T_1 - the time at which the ball leaves the racquet

P_2 - the position at which the ball touches the ground on the other side of the court

T_1 - the time at which the ball touches the ground on the other side of the court.”

The average speed is

$$s = \frac{P_2 - P_1}{T_2 - T_1}$$

So let's use our new equations to work out the uncertainty in s , and compare it to the uncertainty we got from the simple method above.

We have two subtractions and a division here. Use the equations to work out the uncertainties in each bit first.

So:

$$s = \frac{A}{B}$$

where $A = P_2 - P_1$ and $B = T_2 - T_1$.

Using the equation for adding or subtracting numbers, we find that

$$\sigma_A = \sqrt{\sigma_{P_2}^2 + \sigma_{P_1}^2} \text{ and}$$

$$\sigma_B = \sqrt{\sigma_{T_2}^2 + \sigma_{T_1}^2}.$$

Using the equation for dividing numbers, we find that

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2.$$

Putting this all together,

$$\left(\frac{\sigma_s}{s}\right)^2 = \frac{\sigma_{P_2}^2 + \sigma_{P_1}^2}{(P_2 - P_1)^2} + \frac{\sigma_{T_2}^2 + \sigma_{T_1}^2}{(T_2 - T_1)^2}$$

We know that:

$$P_1 = 0.5 \pm 0.1m$$

$$P_2 = 22.1 \pm 0.1m$$

$$T_1 = 1.0 \pm 0.05s$$

$$T_2 = 2.3 \pm 0.05s$$

And the average speed s is 16.62 m s^{-1} .

so

$$\sigma_{P_2} = \sigma_{P_1} = 0.1m$$

$$\sigma_{T_2} = \sigma_{T_1} = 0.05s$$

$$P_2 - P_1 = 21.6m \text{ (use the average value)}$$

$$T_2 - T_1 = 1.3s \text{ (use the average value again).}$$

Substitute these values in to the above equation and we find that:

$$\left(\frac{\sigma_s}{16.62}\right)^2 = \frac{0.1^2 + 0.1^2}{21.6^2} + \frac{0.05^2 + 0.05^2}{1.3^2} = 0.0000428 + 0.00295$$

Note that the second term on the right hand side is MUCH greater than the first. This tells us that the timing uncertainty is much more serious than the distance uncertainty.

Take the square root of both sides and multiply by 16.62 and you find that $\sigma_s = 0.91ms^{-1}$.

So using the sophisticated method, we find that the speed s is:

$$s = 16.62 \pm 0.91$$

, while using the simpler method, we found

$$s = 16.62_{-1.34}^{+1.55}ms^{-1}$$

So the more sophisticated method gives s a somewhat smaller uncertainty (because it assumes that the uncertainties in the two positions and times will often have opposite effects on the result and hence partially cancel out).