Angular Momentum

A vector...

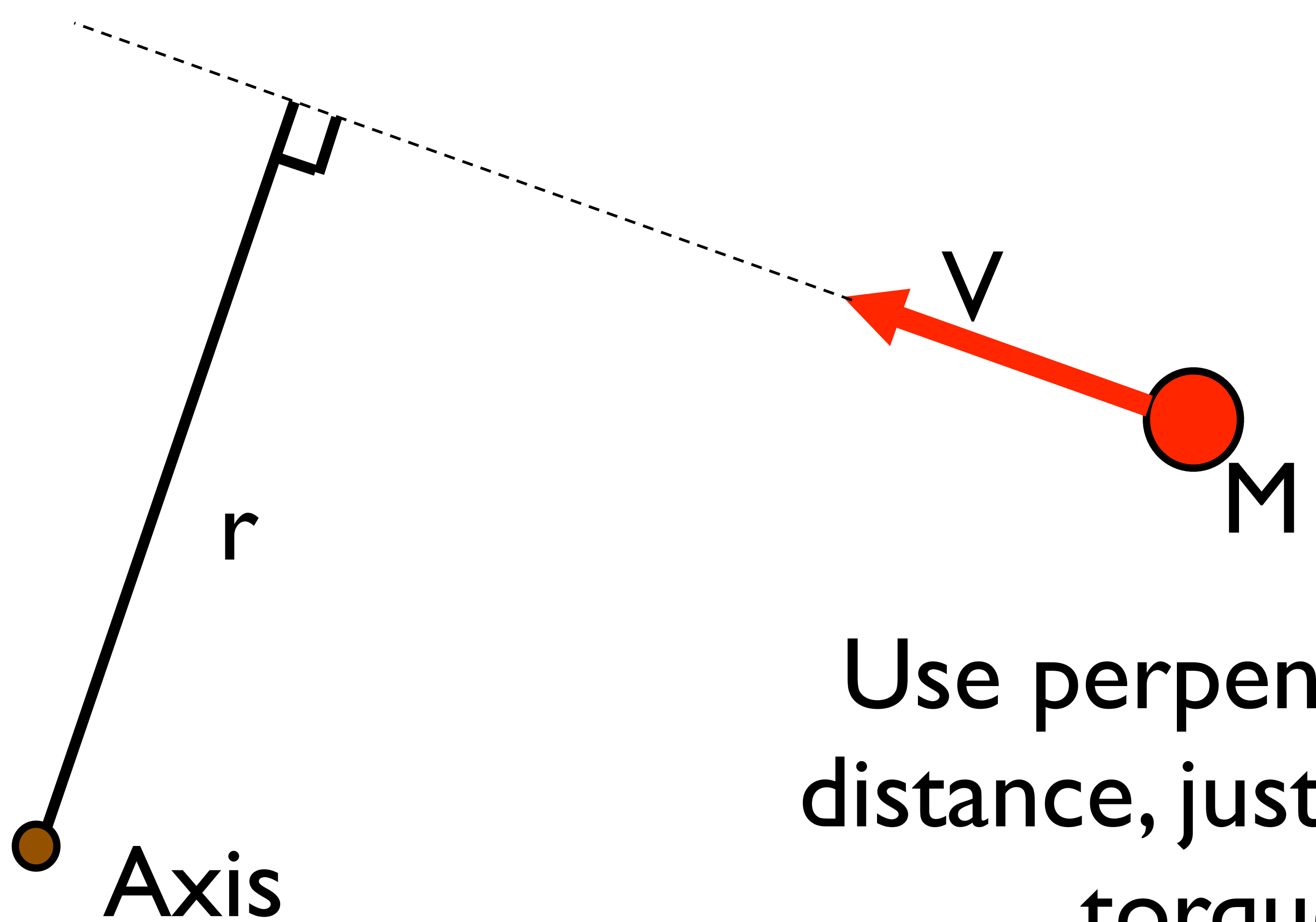
Clickers Channel D

Analogy - between forces and torques.

- Can we take this further and define an angular equivalent of momentum?

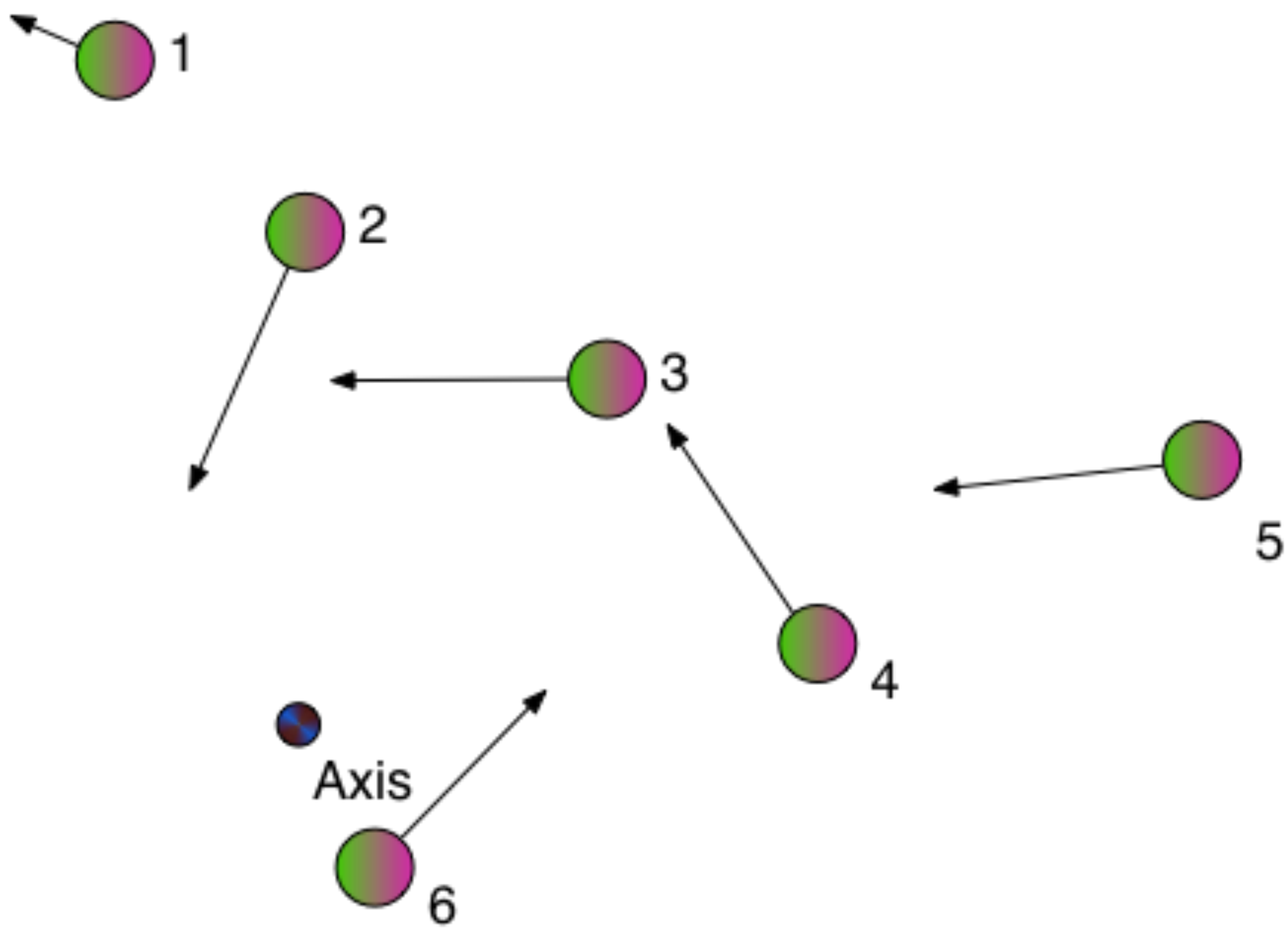
Angular Momentum

- **Angular momentum:** written L
  - Definition   L = M V r

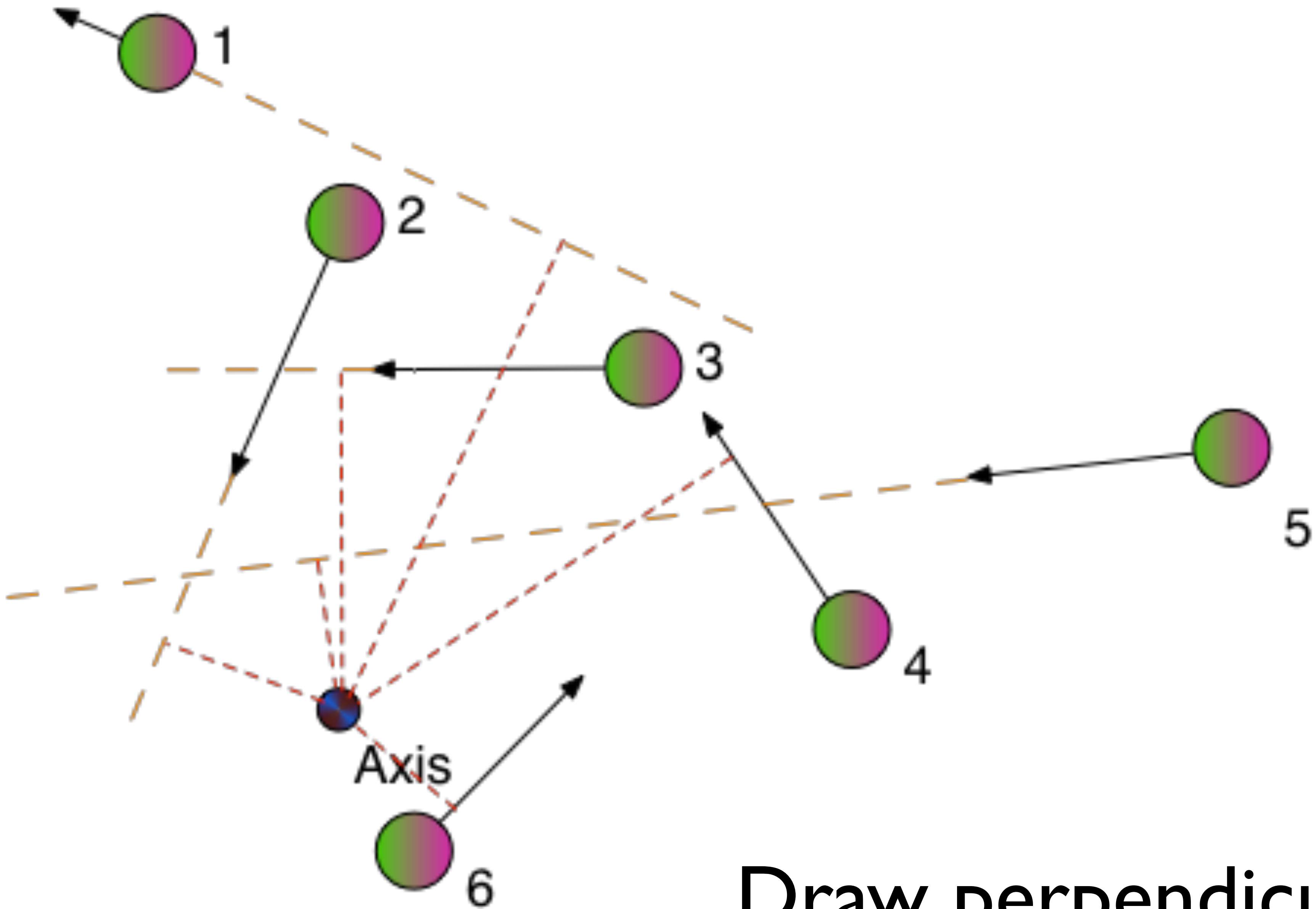


Use perpendicular distance, just like for torque

Which ball has the most angular momentum?



Answer



Draw perpendicular lines.

1 is furthest but its speed is less.

All the others have the same speed

- 4 is furthest of them

Or for a solid rotating object

- $L = I \omega$ (moment of inertia around whatever the axis is times angular velocity)

Orbit simulation

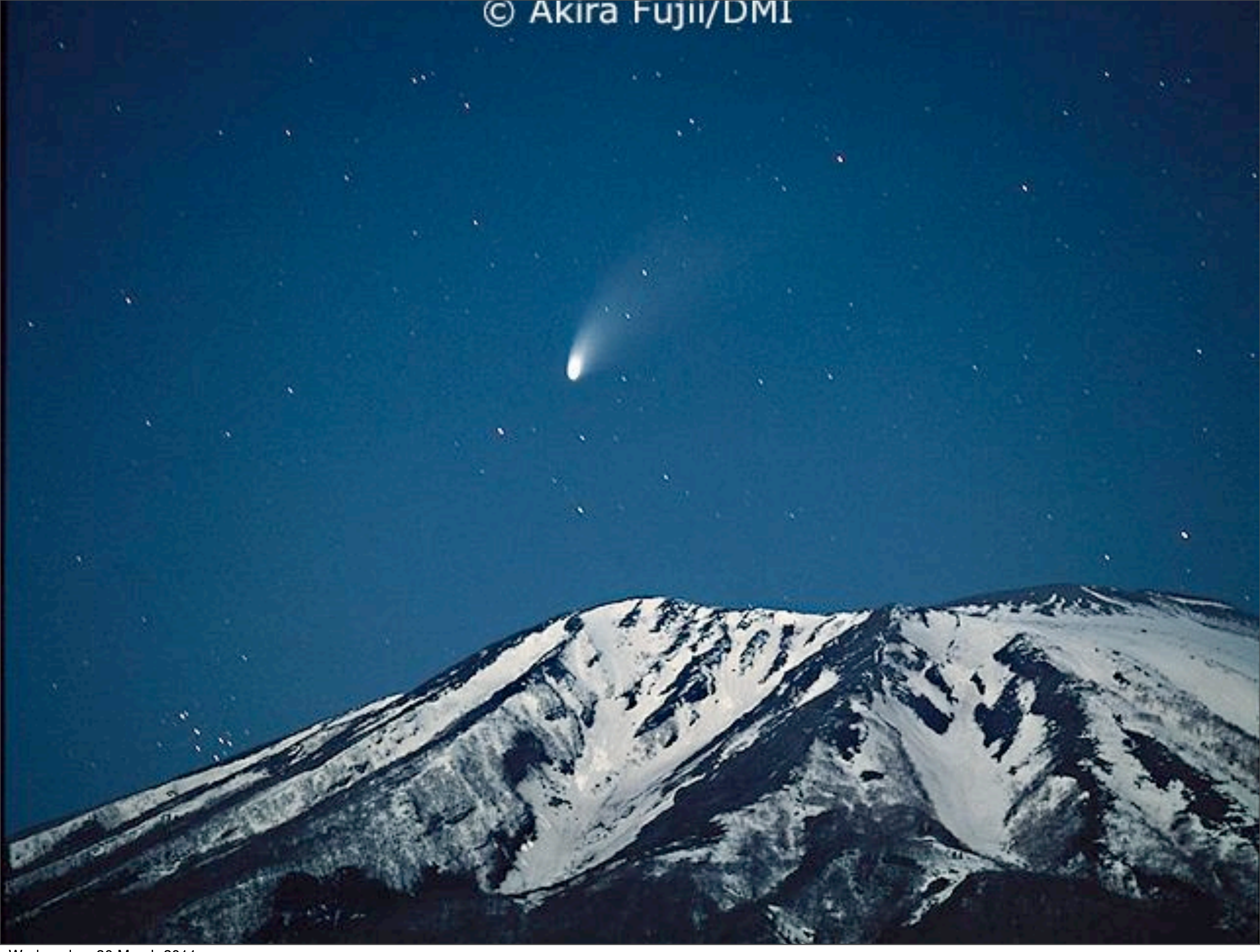
- Due to conservation of angular momentum, an orbiting body goes faster as it gets closer in.
- Makes it very hard to feed black holes!

How do you calculate with this?

- Write down the angular momentum at the start and end, set them equal and solve.

Example

- A typical long-period comet passes perihelion (it's closest point to the Sun) one Astronomical Unit from the Sun, traveling at 70 km/sec.
- How fast would it go when at its furthest point from the sun (aphelion), 50,000 Astronomical Units out?
- (An astronomical unit is the distance of the Earth from the Sun - 1.5×10^{11}m).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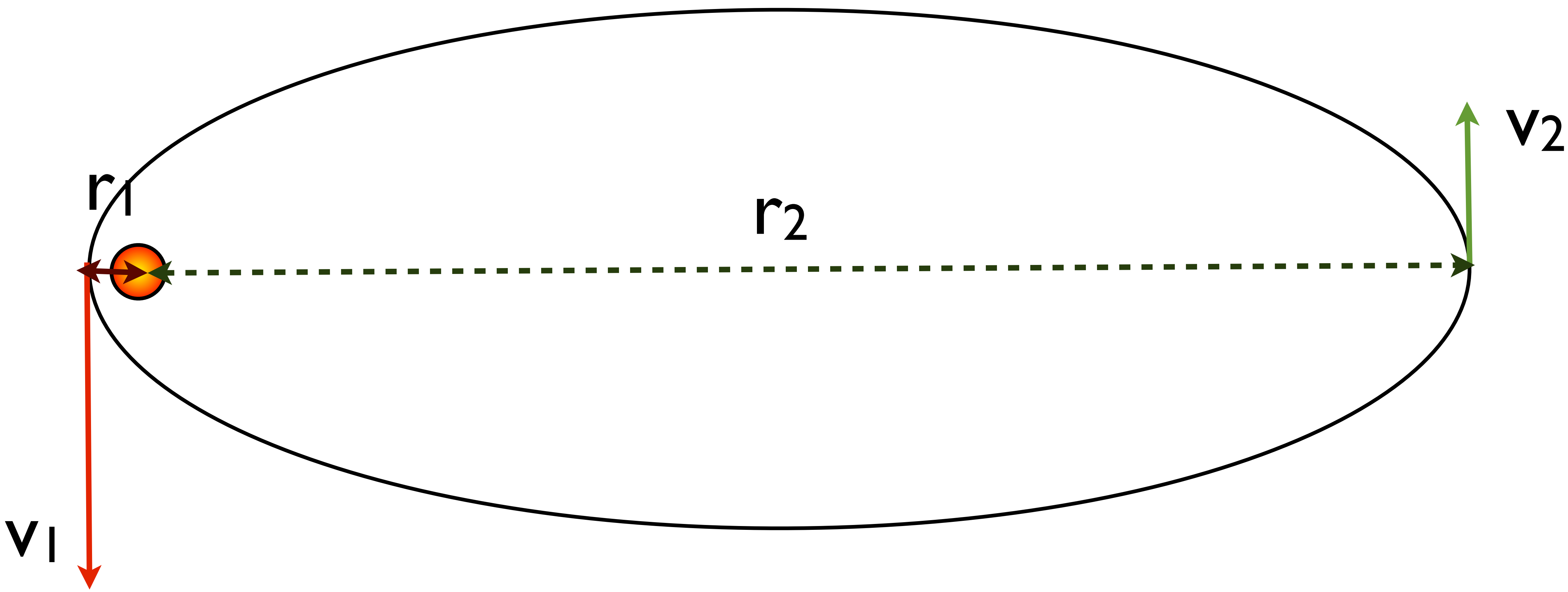
Conserve Angular Momentum

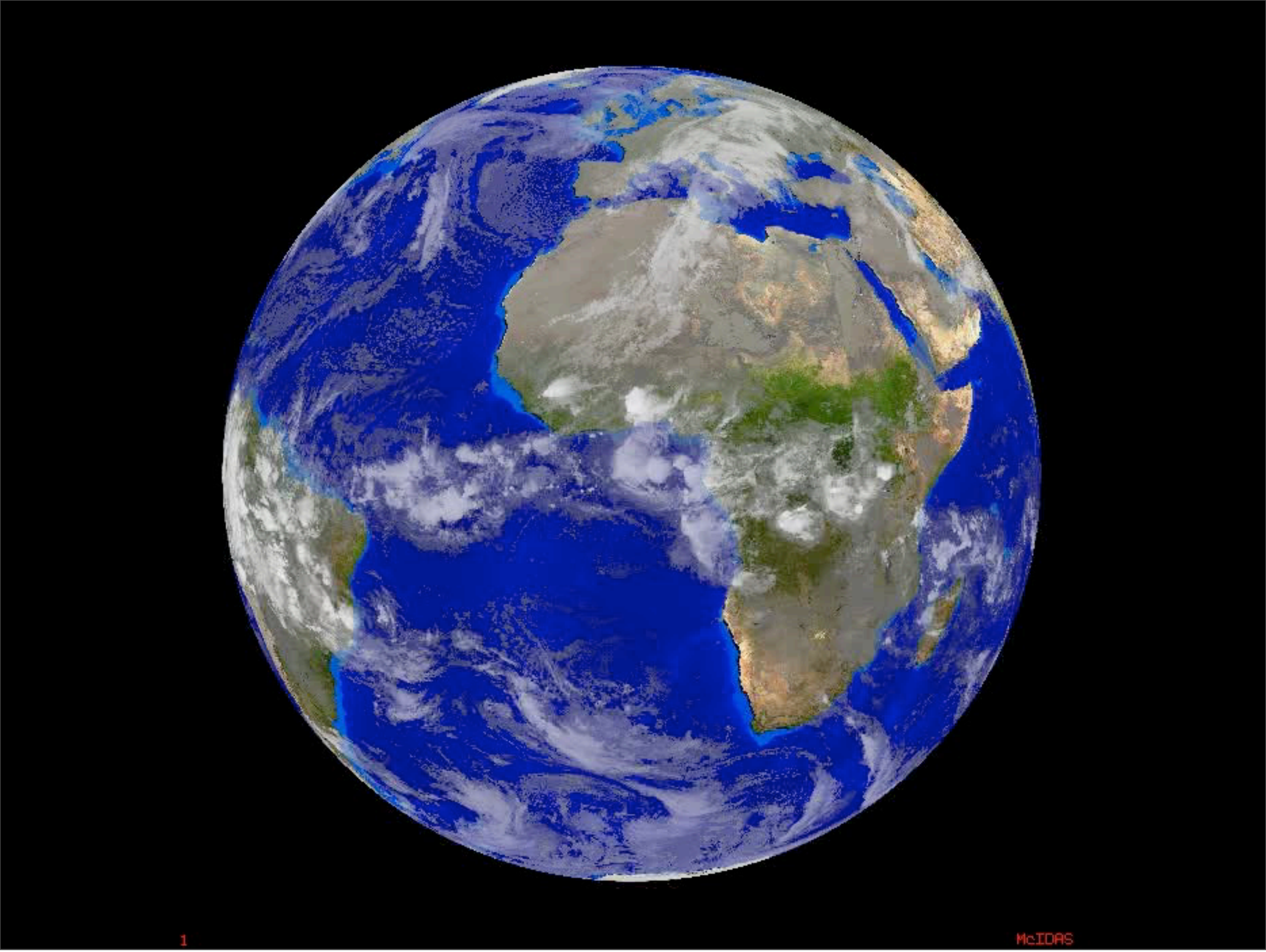


$$mr_1v_1 = mr_2v_2$$

$$v_2 = \frac{r_1}{r_2}v_1 = \left(\frac{1}{50,000}\right)70,000 = 1.4m/s$$

If something gets smaller...

- It rotates faster to compensate!
- Demonstration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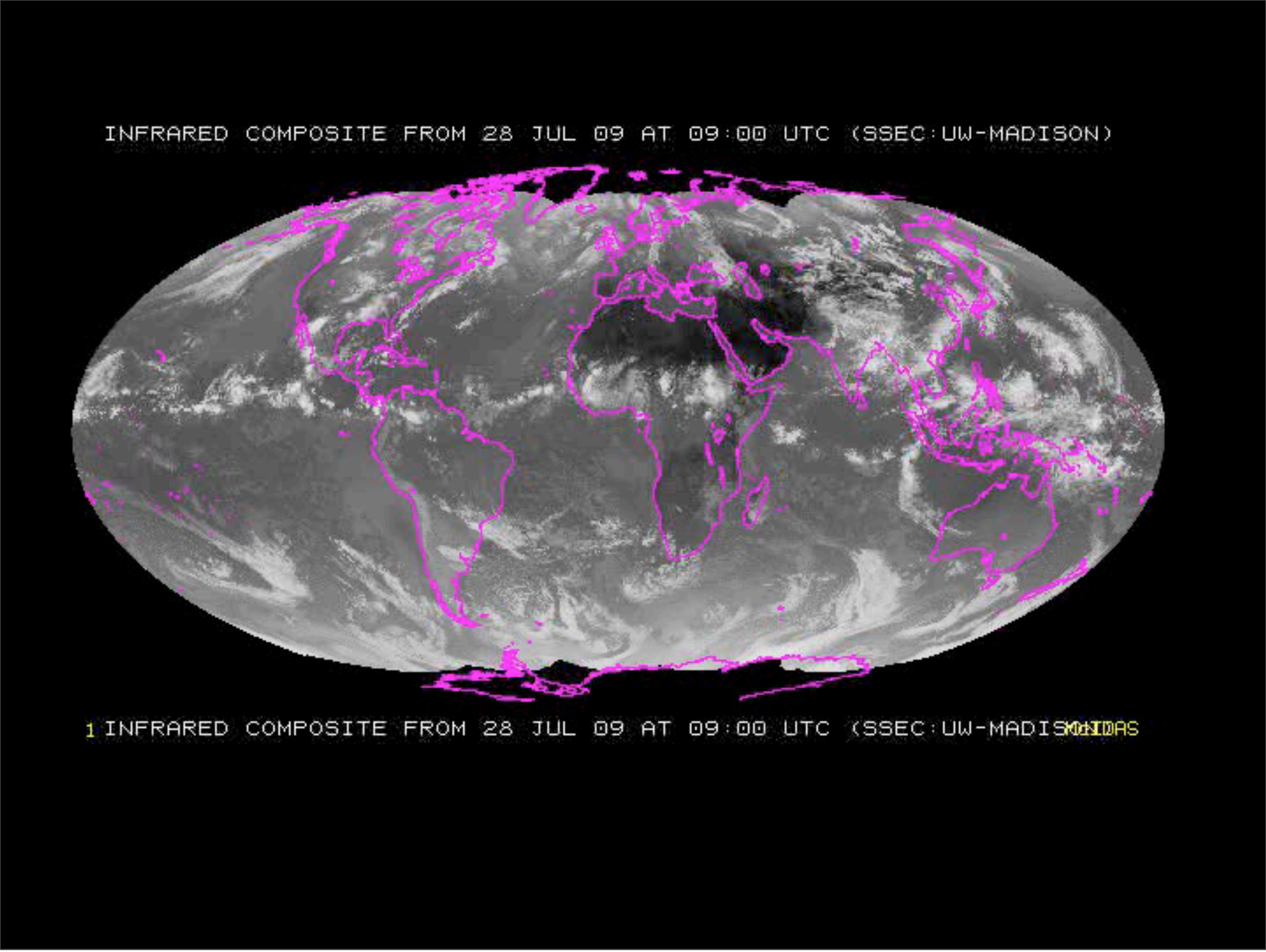
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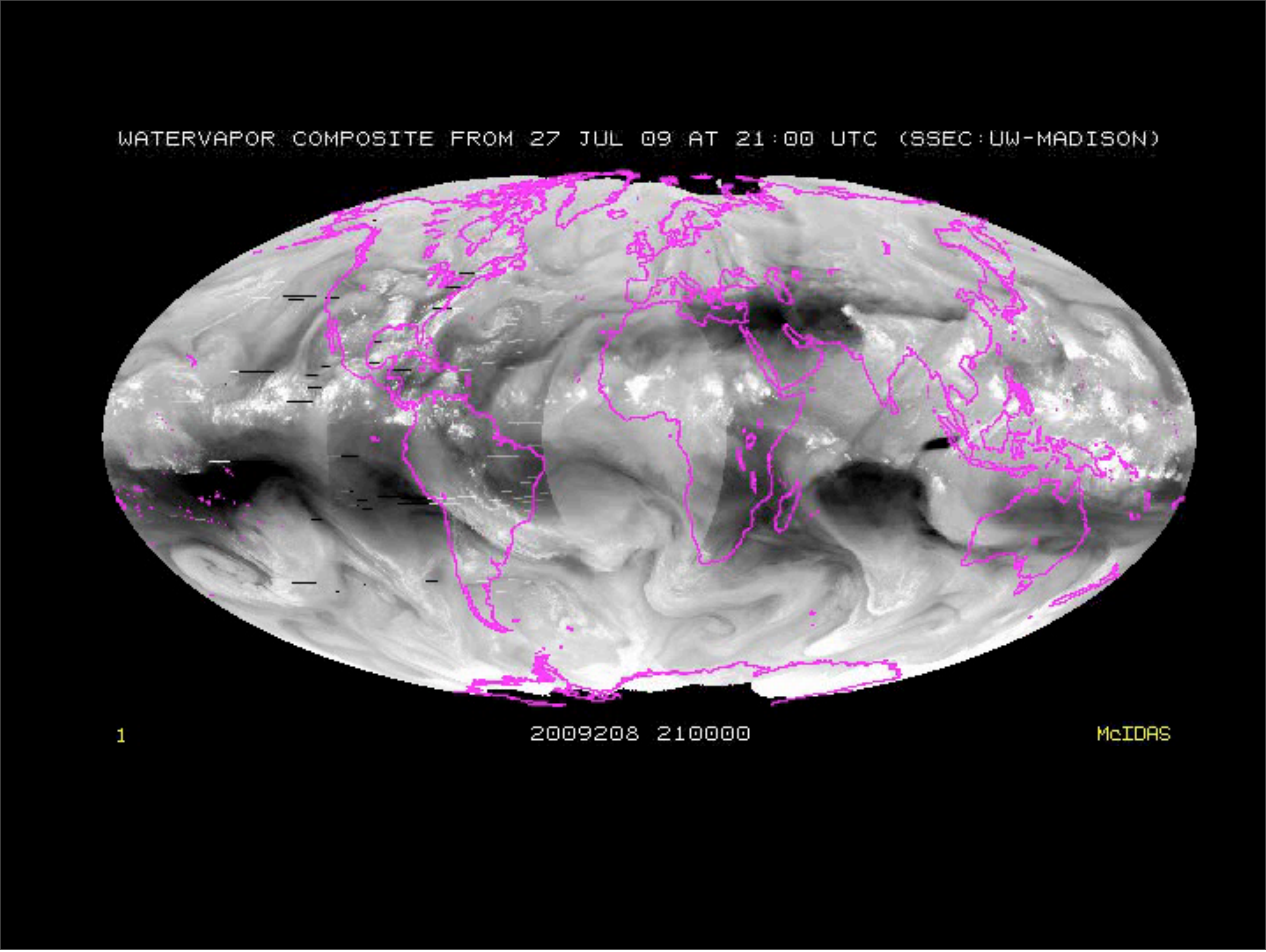
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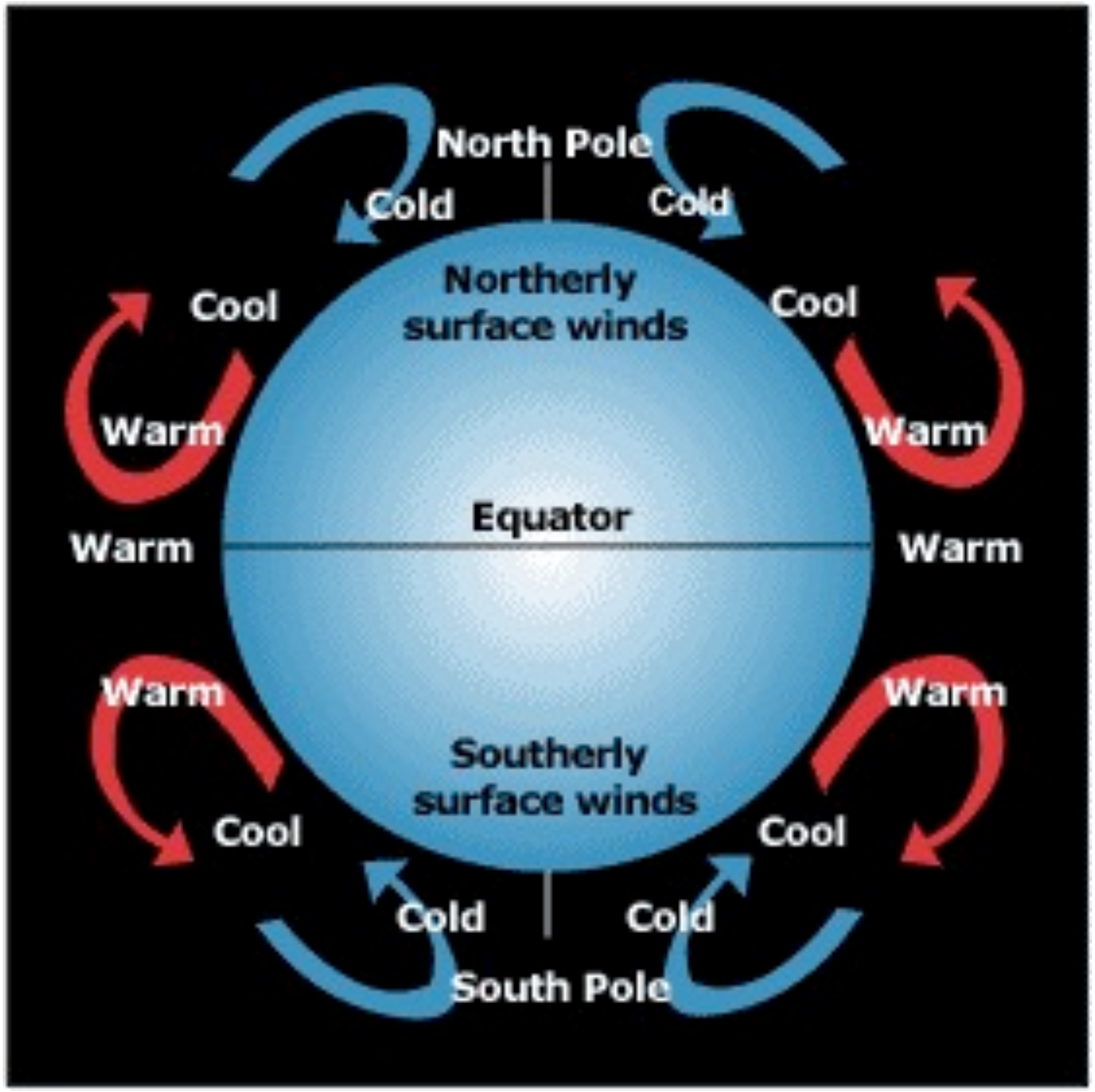
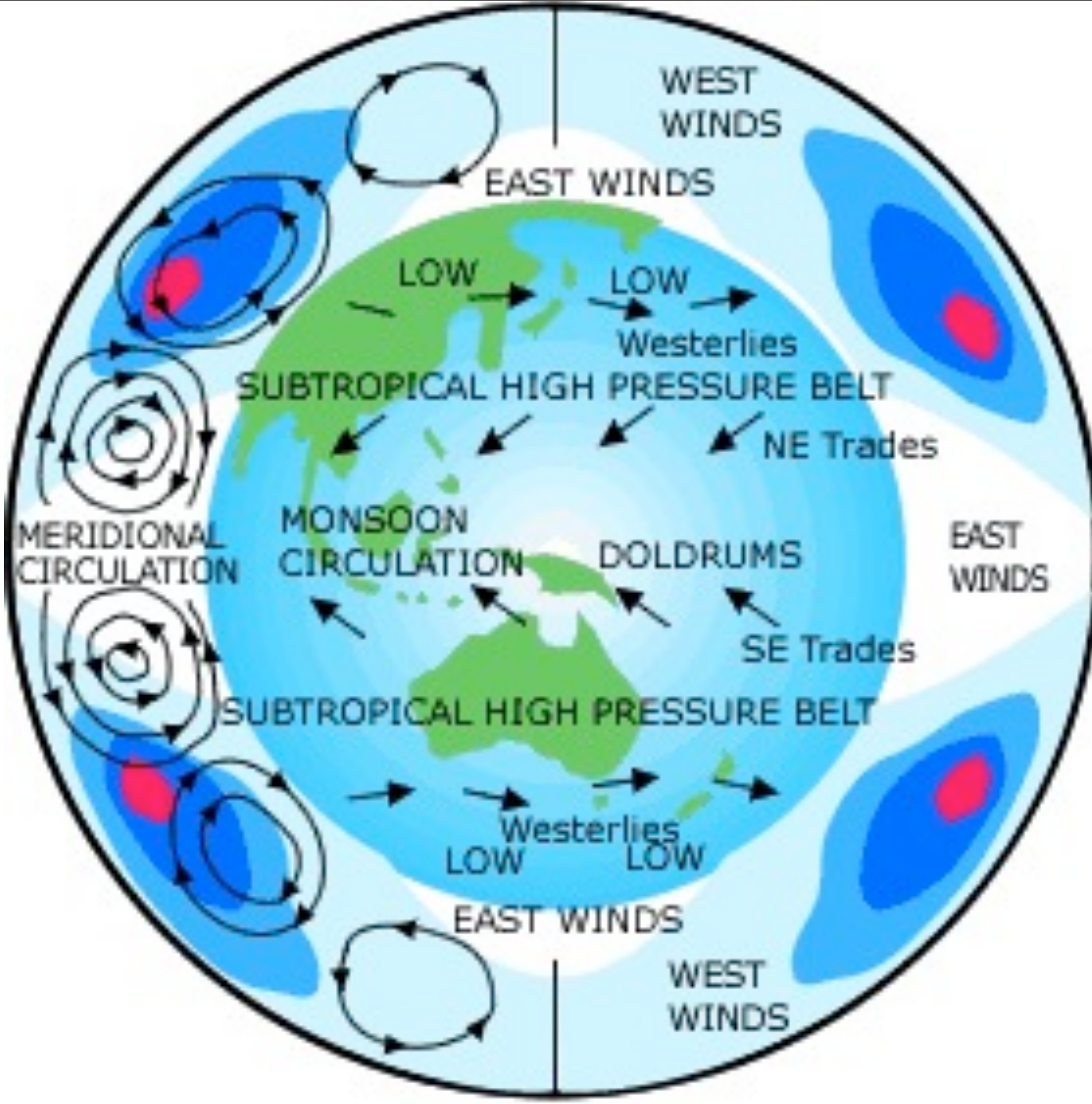
Very distinctive patterns!

- Why?
- All driven by the fact that hot air rises.
- Especially if it is very humid (as water vapor condenses as it rises, liberating heat and keeping the rising going).
- This is most powerful near the equator, as that's where most heat arrives.

So air rises at the equator

- It must go somewhere (law of conservation of “stuff”).
- It flows outwards at high altitudes until it cools and falls.
- These are known as Hadley Cells




WEST WINDS
EAST WINDS
LOW
LOW
Westerlies
SUBTROPICAL HIGH PRESSURE BELT
NE Trades
MERIDIONAL CIRCULATION
MONSOON CIRCULATION
DOLDRUMS
EAST WINDS
SE Trades
SUBTROPICAL HIGH PRESSURE BELT
Westerlies
LOW
LOW
EAST WINDS
WEST WINDS

But the air does not just move north and south

- Due to conservation of angular momentum!
- If moving away from the poles, its distance from the Earth's radius gets less.
- So its velocity must get greater to compensate…

Torque, Angular momentum...

- So far, we've treated them as scalars.
- But - remember the analogy, normal momentum and force (their counterparts) are vectors.
- Could things like torque, angular momentum and angular velocity ω be vectors too?

A ship that didn’t work…

- D/V Chikyu
- A half billion US dollar research vessel designed to drill 7 km into the crust of the Earth beneath the oceans, and sample the Earth’s mantle directly.

It has to hold its place

- While drilling into the sea floor, 4km below.
- A whole set of huge swivelling propellers are used to keep it in place, against the buffeting of tides, waves and wind.
- And that is where the problem comes in…
- The engineers had obviously forgotten what I'm going to show you today…

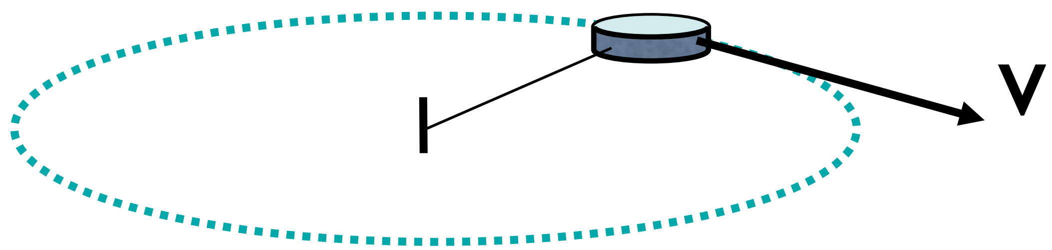
Demonstration

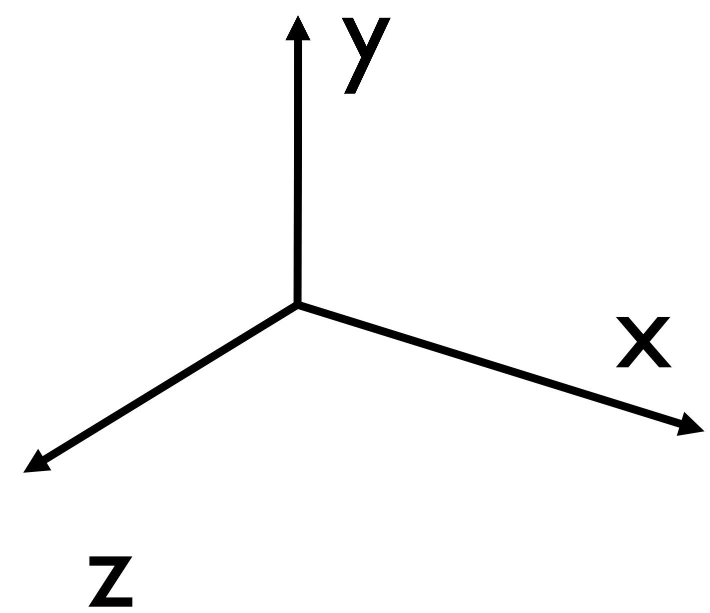
Vectors!

- It looks like the DIRECTION in which something is spinning is important.
- Things seem to like to keep spinning in the same direction.
- If you try to push them in some other direction, they resist - and worse, they do funny things.
- This all sounds like VECTORS…

Very important

- It turns out that this little-appreciated fact (it wasn't discovered 'till the mid 19th century - more than 100 years after Newton codified angular momentum) is incredibly important.
- It explains why the D/V Chikyu doesn't work well
- It explains how brain scanners work
- It allows rockets to navigate
- It decided both world wars
- It explains why you can ride a bike
- It explains why there is a North pole but not an East pole
- It even explains why horoscopes are crap.





- Consider the uniformly rotating object shown above. If the object's angular velocity is a vector, is there a particular direction we should associate with the angular velocity?

1. Yes, ±x
2. Yes, ±y
3. Yes, ±z
4. Yes, some other direction
5. No - the choice is really arbitrary

But I am constant as the northern star,

Of whose true-fix'd and resting quality

There is no fellow in the firmament.

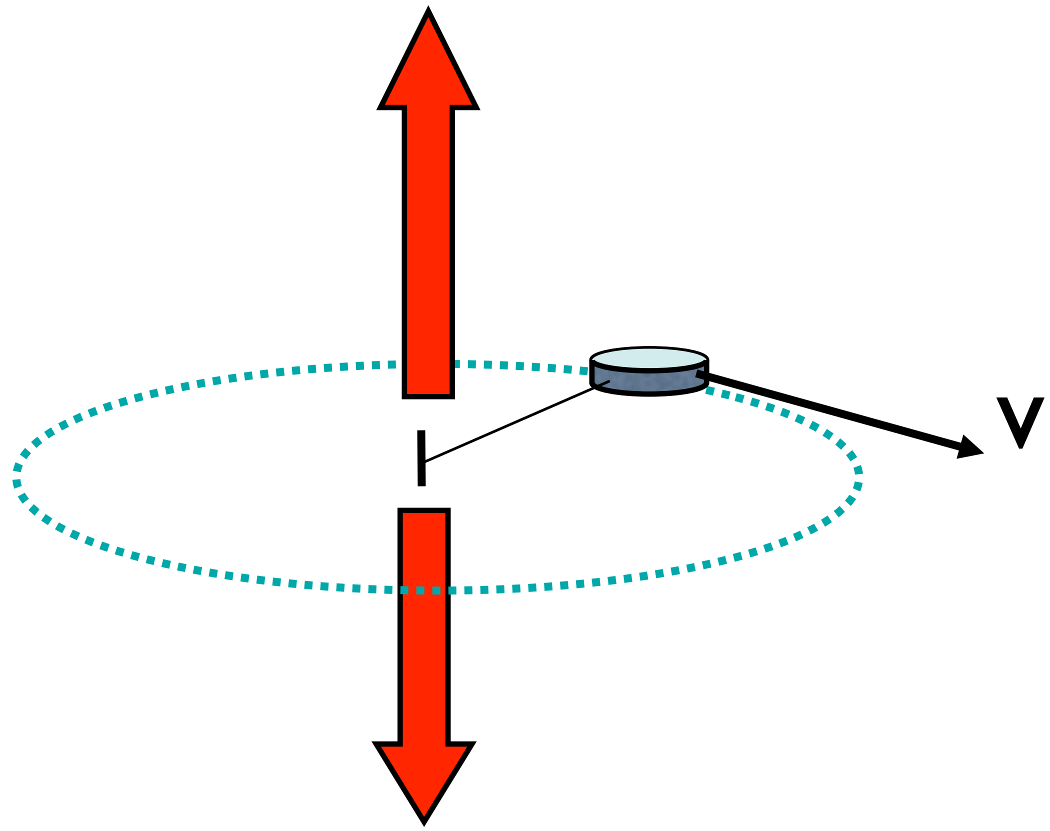
The skies are painted with unnumber'd sparks,

They are all fire and every one doth shine,

But there's but one in all doth hold his place:

(Shakespeare, Julius Caesar)

Perpendicular vector!



- The only sensible vector is one along the axis of rotation.
- Perpendicular to the plane in which the rotation takes place.
- The velocity vector would not be a sensible choice as it is constantly changing.
- Only the perpendicular vector remains constant.
- In principle could go either up or down

Definition of Angular Velocity (full, vector form)

- Direction is (by convention) obtained from the right-hand rule.
- Four fingers point in the direction of rotation.
- Thumb points along axis of rotation in direction of angular velocity.
- Magnitude of the angular velocity vector is equal to the rotation speed in radians per second.

And vector angular momentum

- Points in the same direction as the angular velocity.
- Magnitude is Iω, or for a small moving object, mrv, where r is the perpendicular distance discussed before.

Cross Product

$$\overline{L} = \overline{r} \times \overline{p}$$

- ***r*** is the vector from the axis to the object, and ***p*** is the momentum vector of the object.
  - The x indicates a "cross product"

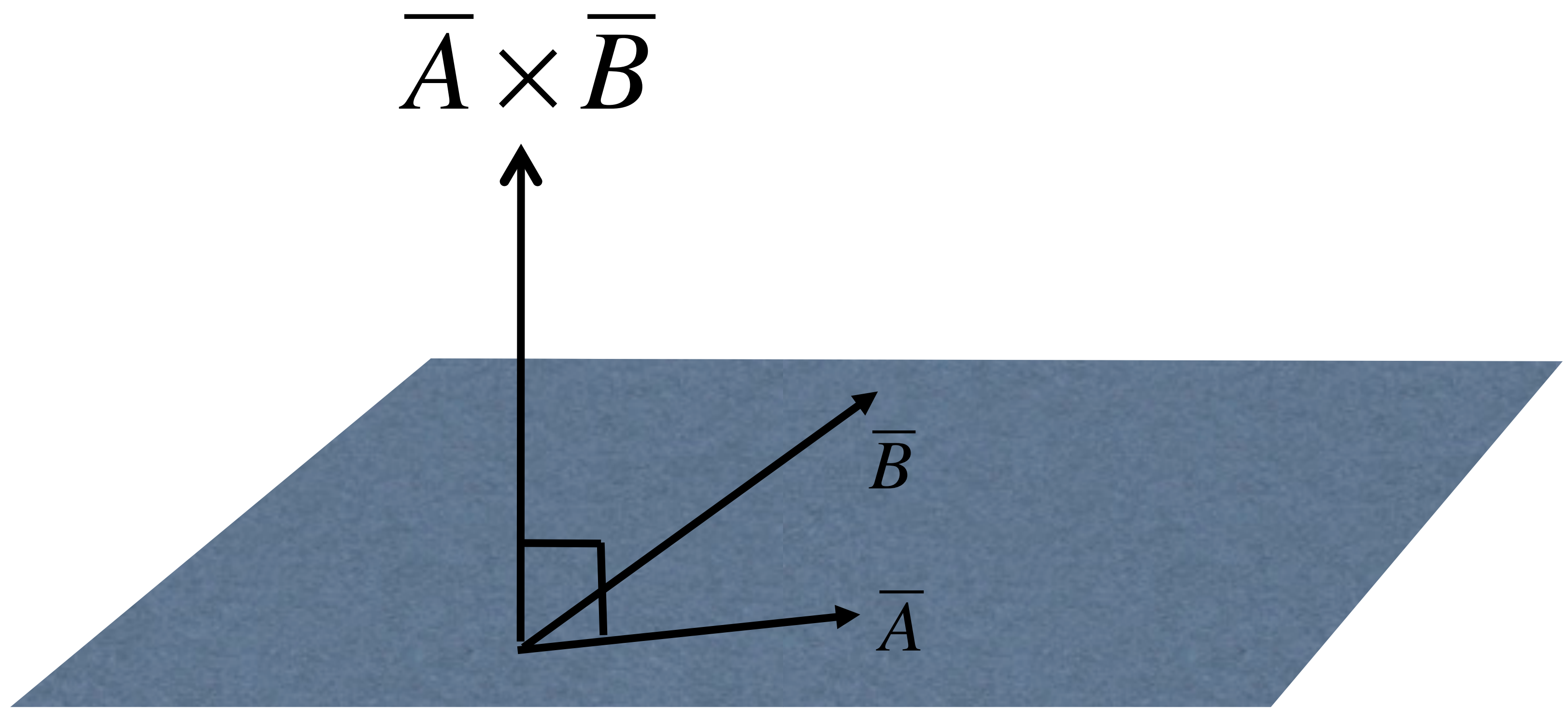
Cross product of two vectors A and B

- If A = (A_x,A_y,A_z) and B = (B_x,B_y,B_z), then

$$\overline{A} \times \overline{B} = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

- (note the symmetry in the pattern)

Or in geometrical terms…



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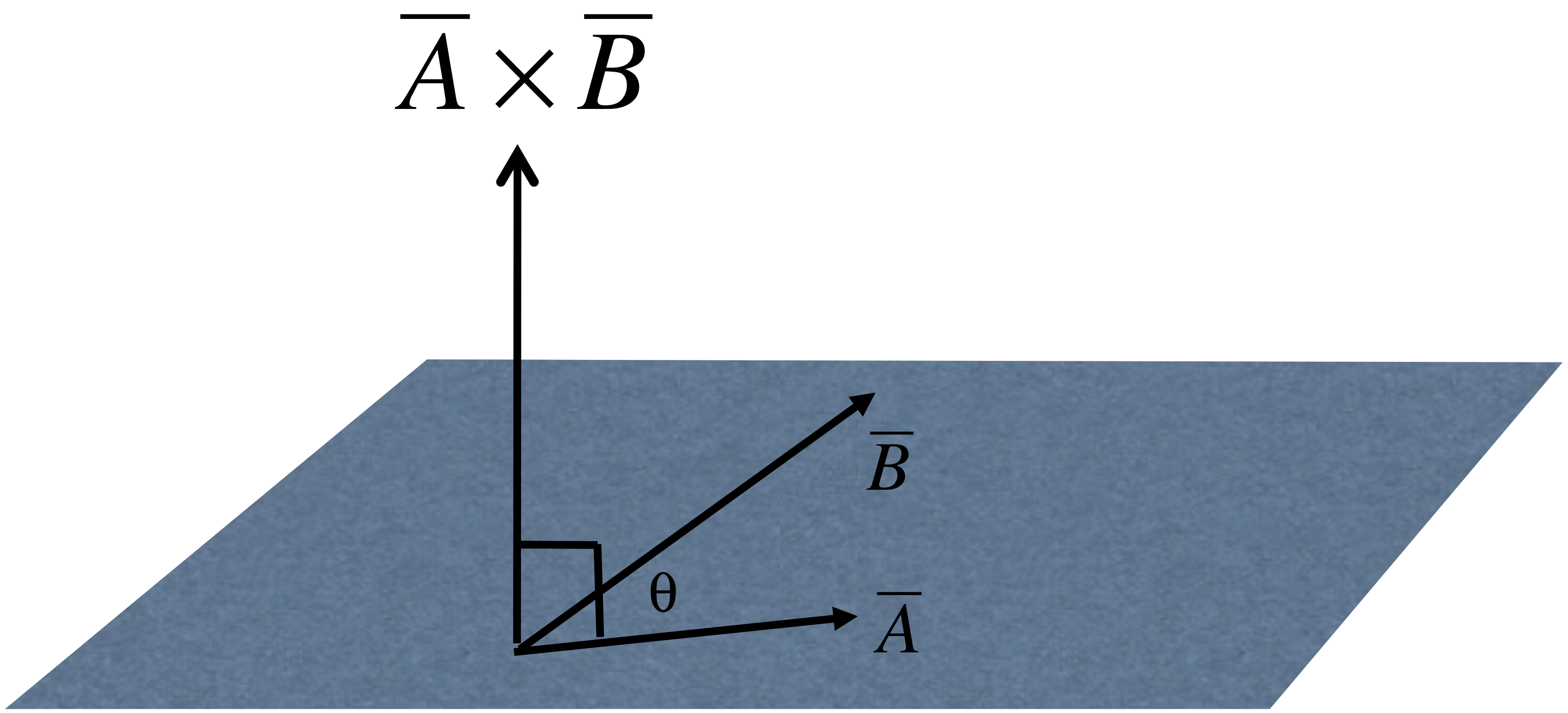
Magnitude of vector is $$\left|\overline{A} \times \overline{B}\right| = \left|\overline{A}\right|\left|\overline{B}\right|\sin(\theta)$$

Definition of Angular Velocity (full, vector form)



- Right hand rule applies here too: rotate four fingers from A to B (first to second term), then the thumb will point in the right direction.
- Note that AxB is the reverse of BxA. The order matters

VPython simulation

Torque is also a vector

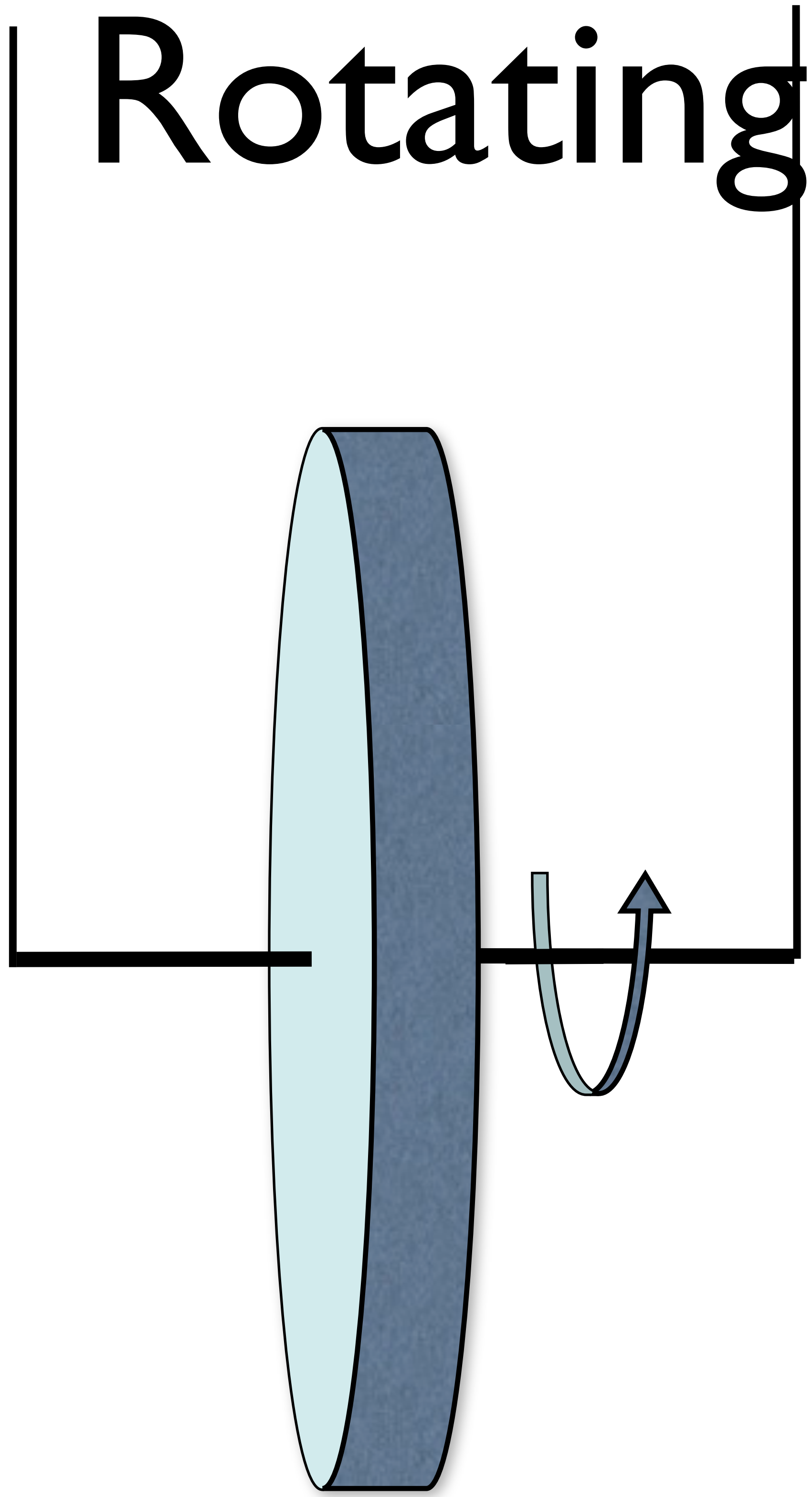
$$\bar{\tau} \equiv \bar{r} \times \bar{F}_{net}$$

- As torque is the rate of change of angular momentum (a vector), it too must be a vector.
- Once again, it uses the cross product, and is perpendicular to the plane including r and F_{net}.

Weird motion of the bike wheel

- When you try to turn it one way, it moves off at right-angles!
- Why?

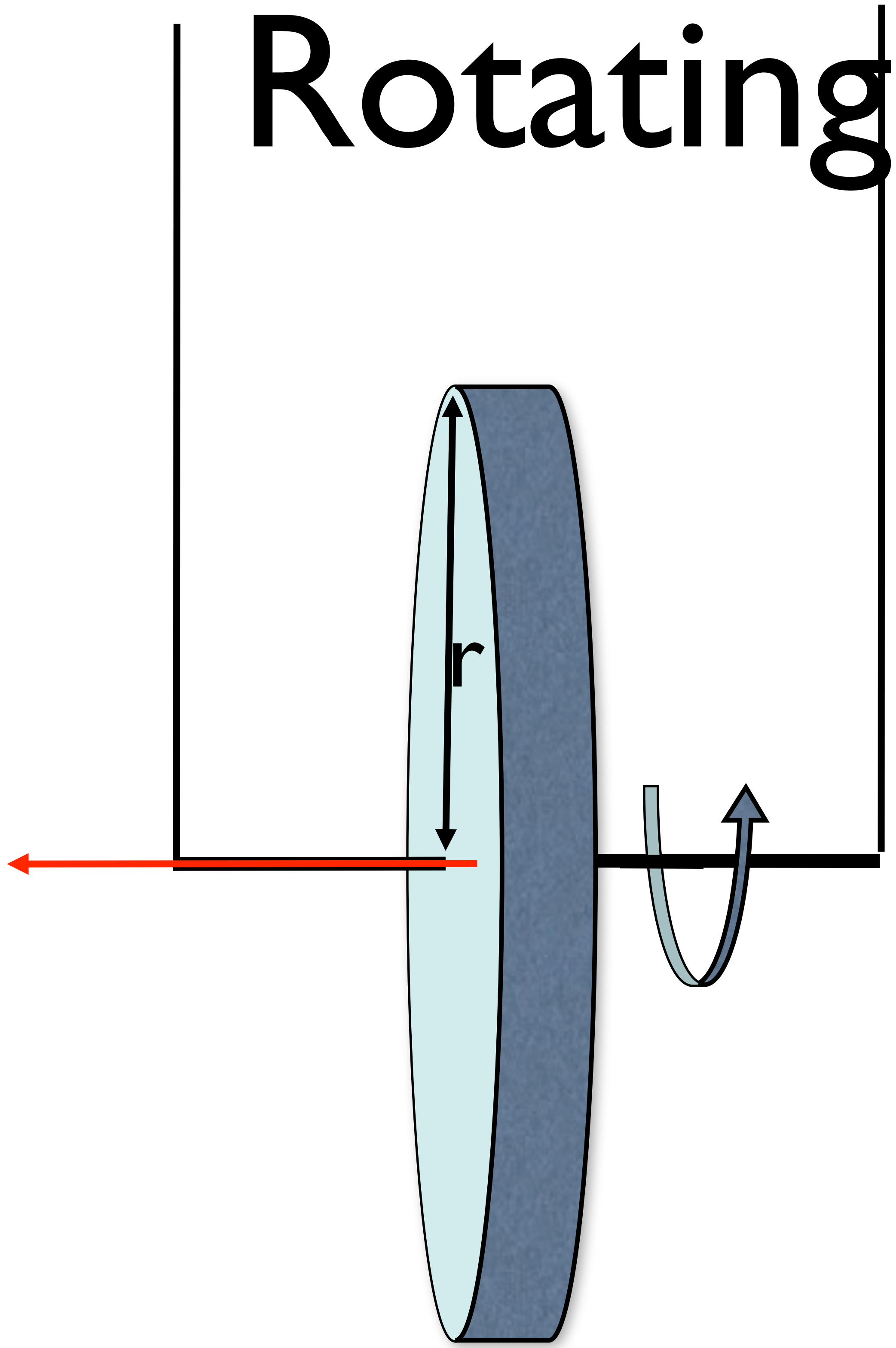
Rotating wheel



- Supported by two cables.
- In which direction does its angular momentum point?

Clicker Question

Rotating wheel



- By the right-hand rule, the angular momentum points out to the left.

$$L = I\omega = \tfrac{1}{2} m r^2 \omega$$