Full derivation of the projectile motion equations

Acceleration is defined as the rate of change of velocity. So, by definition...

$$a = \frac{dv}{dt}$$

For the projectile motion case, acceleration is constant. So what is the velocity? We know dv/dt and we want to know v. This means undoing the differentiation. To undo differentiation, you need to integrate. So lets integrate both sides of the equation.

$$\int a\ dt = \int \frac{dv}{dt} dt$$

The right hand is just the integral of the differential of v - i.e. it takes you back to v. For the left hand side, to integrate a constant, multiply it by whatever variable you are integrating against (in this case, t) and add a constant. So we get

$$at + C = v$$

where C is a constant (whenever you do an indefinite integral, you end up with a constant).

As usual with constants like this, you work out their value by setting t = 0. In this case, v=C. So C is just the velocity at time zero. Which can be written as v_0.

Now, let's try to work out the position s. By definition, velocity is the rate of change of position, i.e.

$$\frac{ds}{dt} = v$$

Once again, we need to reverse the differentiation, so we need to integrate both sides of the equation.

$$\int \frac{ds}{dt} dt = \int v dt = \int (at + v_0) dt$$

The left hand term is just s (differentiating it then integrating it just takes you back to where you started).

The right-most part is just substituting the above equation for velocity in. So we need to integrate at + v_0 with respect to t.

A is a constant. t is raised to the power 1, so its integral is t raised to the power 2, with a constant of 1/2 in front (usual law for integrating polynomials). v_0 becomes v_0t, and a new constant is needed (you always add a constant when you do an indefinite integral).

$$s = \frac{1}{2} a t^2 + v_0 t + K$$

I've written the constant as K rather than C, to make sure it doesn't get confused with the constant C above.

As usual you work out the constant by setting the time to zero, in which case you find that s+K, i.e. K is the position at time zero, which I will write S_0. So this becomes

$$s - s_0 = \frac{1}{2} a t^2 + v_0 t$$

I hope this makes more sense!